Unit D4

Continuity

Introduction

In Unit A4 *Real functions, graphs and conics* you studied techniques for sketching the graphs of many common real functions. For this purpose, we made various assumptions about these graphs – in particular, that they have no gaps (except 'obvious' ones at asymptotes). In this unit you will see how to justify this assumption by using the concept of a *continuous function* and showing that many familiar functions are continuous. This concept is important in analysis because, in many cases, the easiest way to *prove* that a function has a property which may seem intuitively obvious is to use the fact that the function is continuous.

1 Operations on functions

This section gives an overview of the various fundamental operations on functions: forming combinations, composites and inverses of functions. These operations were discussed in more detail in Book A *Introduction*.

We begin with a brief review of notation and basic terminology. In this unit we are concerned with *real functions*, that is, functions whose domain and codomain are subsets of $\mathbb{R}$. Such functions can be specified in various ways. For example, the function

$$\begin{aligned} f : \mathbb{R} - \{0\} &\longrightarrow \mathbb{R} \\ x &\longmapsto 1/x \end{aligned}$$

can also be written as

$$f(x) = 1/x \quad (x \in \mathbb{R} - \{0\}),$$

where the codomain of f is assumed to be $\mathbb{R}$. It can also be written simply as

$$f(x) = 1/x,$$

where now the domain of f is assumed to be the set of values of x for which $1/x$ is defined, that is, $\mathbb{R} - \{0\}$, and where the codomain is $\mathbb{R}$. These notations illustrate the following convention, which you met in Unit A4.

Convention for real functions

When a real function is specified *only by a rule*, it is to be understood that the domain of the function is the set of all real numbers for which the rule is applicable, and the codomain of the function is $\mathbb{R}$.

Recall that, if A and B are subsets of $\mathbb{R}$ and $f : A \longrightarrow B$ is a real function, then

- the **image set** of f is the set $f(A) = \{f(x) : x \in A\}$
- f is **one-to-one** if each element of $f(A)$ is the image of exactly one element of A
- f is **onto** if $f(A) = B$.

The above convention for real functions is concise and we often use it. Sometimes, however, if we want to assert that a function has a particular property, then we may need to restrict its domain or codomain. For example, the function

$$f(x) = \sin x$$

has domain and codomain $\mathbb{R}$, by our convention, but it is neither one-to-one nor onto. However, the function

$$\begin{aligned} g : [-\pi/2, \pi/2] &\longrightarrow [-1, 1] \\ x &\longmapsto \sin x \end{aligned}$$

has the same rule as the above function f, but g is both one-to-one and onto.

When we say that a function f is *defined on* a set I (usually an interval), this means that the domain of f contains the set I. For example, the function $f(x) = 1/x$ is defined on $[1, 2]$, but not on $[-1, 1]$. The definitions and notation for the various types of interval (open, closed, half-open) are given in the module Handbook; we make frequent use of them in this unit.

1.1 Sums, products and quotients of functions

Let f and g be the functions

$$f(x) = 1/x \quad (x \in \mathbb{R} - \{0\}) \quad \text{and} \quad g(x) = \sin x \quad (x \in \mathbb{R}).$$

The graphs of these functions are shown in Figure 1.

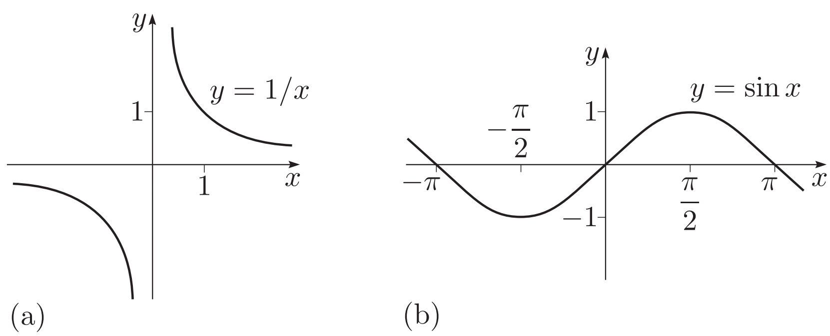


Figure 1 The graphs of (a) $y = 1/x$ and (b) $y = \sin x$

We use the expressions $f+g$, fg and f/g to denote the functions

$$(f+g)(x) = f(x) + g(x) = 1/x + \sin x \quad (x \in \mathbb{R} - \{0\}),$$

$$(fg)(x) = f(x)g(x) = \frac{\sin x}{x} \quad (x \in \mathbb{R} - \{0\})$$

and

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{1}{x \sin x} \quad (x \in \mathbb{R} - \{n\pi : n \in \mathbb{Z}\}).$$

The domains of $f+g$, fg and f/g include only those points at which f and g are *both* defined. (We often use the word 'point' to mean 'number'.) Also, when forming the quotient f/g, we must exclude from the domain all the points x such that $g(x) = 0$. The formal definitions are as follows.

Definitions

Let A and B be subsets of $\mathbb{R}$ and $f : A \longrightarrow \mathbb{R}$ and $g : B \longrightarrow \mathbb{R}$ be functions. Then

- the **sum** $f+g$ is the function with domain $A \cap B$ and rule

  $$(f+g)(x) = f(x) + g(x)$$

- for $\lambda \in \mathbb{R}$, the **multiple** λf is the function with domain A and rule

  $$(\lambda f)(x) = \lambda f(x)$$

- the **product** fg is the function with domain $A \cap B$ and rule

  $$(fg)(x) = f(x)g(x)$$

- the **quotient** f/g is the function with domain

  $$A \cap B - \{x : g(x) = 0\}$$

  and rule

  $$(f/g)(x) = f(x)/g(x).$$

Often we wish to form the sum, product or quotient of functions f and g which have the *same* domain, A say. In this case, A is also the domain of $f+g$ and fg, and the domain of f/g is $A - \{x : g(x) = 0\}$.

Exercise D54

Let f and g be the functions

$$f(x) = e^x \quad (x \in \mathbb{R}) \quad \text{and} \quad g(x) = \tan x \quad (x \in (-\pi/2, \pi/2)).$$

Determine the domain and rule of the functions $f+g$, fg and f/g.

1.2 Composite functions

Let f and g be functions. Then the composite function $g \circ f$ is the function defined by the rule

$$(g \circ f)(x) = g(f(x)),$$

where we apply first f and then g. Again, we must exclude from the domain all the points x which lead to an expression that is not defined.

For example, if

$$f(x) = 1/x \quad (x \in \mathbb{R} - \{0\}) \quad \text{and} \quad g(x) = \sin x \quad (x \in \mathbb{R}),$$

then $g \circ f$ is the function

$$(g \circ f)(x) = \sin \frac{1}{x} \quad (x \in \mathbb{R} - \{0\}),$$

whereas $f \circ g$ is the function

$$(f \circ g)(x) = \frac{1}{\sin x} = \operatorname{cosec} x \quad (x \in \mathbb{R} - \{n\pi : n \in \mathbb{Z}\}).$$

The graphs of $g \circ f$ and $f \circ g$ are shown in Figure 2.

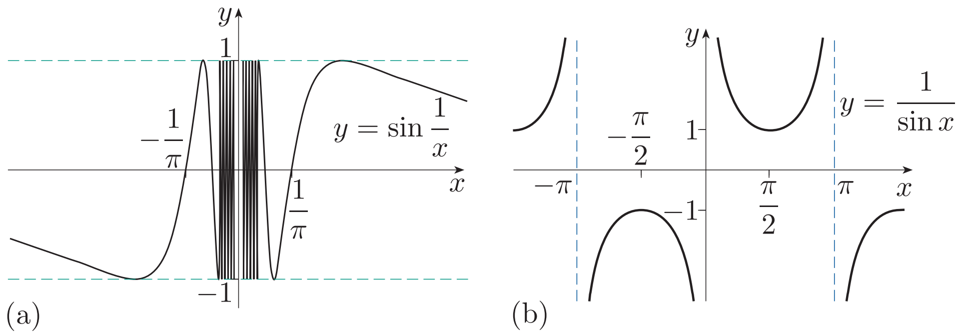


Figure 2 The graphs of (a) $y = \sin \frac{1}{x}$ and (b) $y = \frac{1}{\sin x}$

More generally, if $f : A \longrightarrow \mathbb{R}$ and $g : B \longrightarrow \mathbb{R}$, then $g(f(x))$ is defined if and only if x lies in the domain of f and $f(x)$ lies in the domain of g, as illustrated in Figure 3.

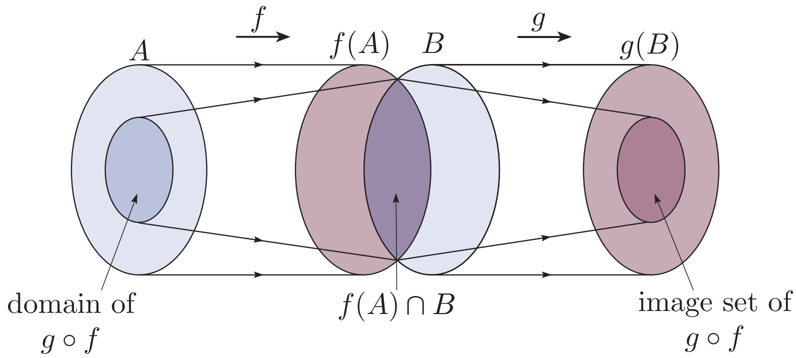


Figure 3 The domain and image set of a composite function

Definition

Let A and B be subsets of $\mathbb{R}$ and $f : A \longrightarrow \mathbb{R}$ and $g : B \longrightarrow \mathbb{R}$ be functions. Then the **composite** $g \circ f$ has domain

$$\{x \in A : f(x) \in B\}$$

and rule

$$(g \circ f)(x) = g(f(x)).$$

This definition allows us to form the composite of *any* two functions, though in some cases the domain of the composite is the empty set $\varnothing$. For example, if

$$f(x) = -x^2 - 1 \quad (x \in \mathbb{R}) \quad \text{and} \quad g(x) = \sqrt{x} \quad (x \in [0, \infty)),$$

then $f(\mathbb{R}) \subset (-\infty, -1]$ which has empty intersection with $[0, \infty)$, the domain of g. So the domain of the composite function $g \circ f$ is $\varnothing$.

Frequently, however, it happens that the image set of f is a subset of the of the domain of g; that is, $f(A) \subseteq B$. In this case, the set A is also the domain of $g \circ f$.

Exercise D55

Let f and g be the functions

$$f(x) = \sqrt{x} \quad (x \in [0, \infty)) \quad \text{and} \quad g(x) = \sin x \quad (x \in \mathbb{R}).$$

Determine the domain and rule of the composites $f \circ g$ and $g \circ f$.

1.3 Inverse functions

Let f be the function

$$f(x) = 2x \quad (x \in \mathbb{R}).$$

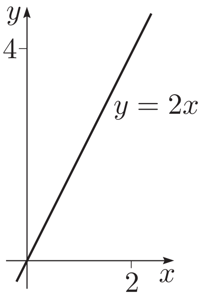


Figure 4 The graph of $y = 2x$

The graph of f is shown in Figure 4. For each number y in $\mathbb{R}$, there is a unique number $x = y/2$ in the domain of f such that

$$f(x) = f\left(\frac{y}{2}\right) = 2 \times \frac{y}{2} = y.$$

The corresponding function $g(y) = y/2$ is called the *inverse function* of f because it undoes the effect of f; that is,

$$g(f(x)) = x, \quad \text{for } x \in \mathbb{R},$$

and

$$f(g(y)) = y, \quad \text{for } y \in \mathbb{R}.$$

However, not every function has an inverse function. For example, consider the function

$$f(x) = x^2 \quad (x \in \mathbb{R}).$$

Since $f(2) = 4 = f(-2)$, we cannot assign a unique value x in the domain of f such that $f(x) = 4$. The problem here is that this function f is not one-to-one. In general, it is possible to define the inverse function of a function only if that function is one-to-one.

We now give the definition of the inverse function, illustrated in Figure 5.

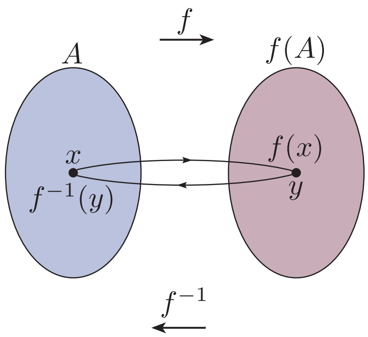


Figure 5 The inverse function

Definition

Let $A \subset \mathbb{R}$ and $f : A \longrightarrow \mathbb{R}$ be a one-to-one function. Then the **inverse function** f^{-1} of f has domain $f(A)$ and rule

$$f^{-1}(y) = x, \quad \text{where } y = f(x).$$

For some functions f, we can find the inverse function f^{-1} directly by solving the equation $y = f(x)$ algebraically to obtain x in terms of y.

Worked Exercise D40

Prove that the following function has an inverse function, and find the domain and rule of this inverse function:

$$f(x) = \frac{1}{1-x} \quad (x \in (-\infty, 1)).$$

Solution

We solve the equation $y = f(x)$ to obtain x in terms of y.

Let

$$y = \frac{1}{1-x}.$$

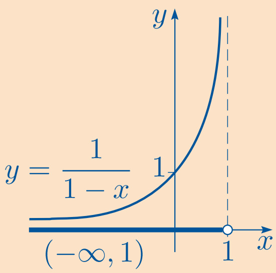
Rearranging this equation, we obtain

$$\begin{aligned} y = \frac{1}{1-x} &\iff \frac{1}{y} = 1 - x \\ &\iff x = 1 - \frac{1}{y}. \end{aligned}$$

This means that each value of y is the image of exactly one value of x, namely $x = 1 - 1/y$.

This shows that f is one-to-one, so f has an inverse function with rule $f^{-1}(y) = 1 - 1/y$.

To determine the domain of f^{-1} we must find the image set of f. A sketch of the graph of f is shown below.



From the graph, it appears that the image set of f is $(0, \infty)$. To prove this, we first show that $f((-\infty, 1))$ is a subset of $(0, \infty)$.

For each x in the domain $(-\infty, 1)$, we have $x < 1$, so

$$f(x) = \frac{1}{1-x} > 0.$$

Thus $f((-\infty, 1)) \subseteq (0, \infty)$.

Next we show that $(0, \infty)$ is a subset of $f((-\infty, 1))$. Taken together, these results show that the image set of f is $(0, \infty)$.

On the other hand, for each y in $(0, \infty)$, we have $1/y > 0$, so

$$x = 1 - \frac{1}{y} \in (-\infty, 1).$$

Thus $f((-\infty, 1)) \supseteq (0, \infty)$, so it follows that $f((-\infty, 1)) = (0, \infty)$.

Hence the domain of f^{-1} is $(0, \infty)$, so

$$f^{-1}(x) = 1 - \frac{1}{x} \quad (x \in (0, \infty)).$$

We have used x instead of y here to conform with the usual practice of writing x for the domain variable when defining a function.

Notice that the graph $y = f^{-1}(x)$ is always obtained by reflecting the graph $y = f(x)$ in the line $y = x$. This is illustrated in Figure 6 for the functions f and f^{-1} from Worked Exercise D40.

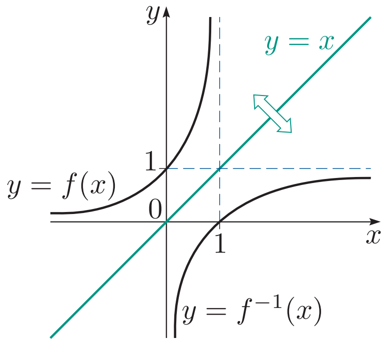


Figure 6 The graphs of the functions in Worked Exercise D40

Exercise D56

Prove that the following function has an inverse function, and find the domain and rule of this inverse function.

$$f(x) = \frac{x+3}{x-2} \quad (x \in (2, \infty))$$

Hint: It may help to write

$$\frac{x+3}{x-2} = 1 + \frac{5}{x-2}.$$

Proving that a function is one-to-one

We have seen that if $f : A \longrightarrow \mathbb{R}$ is one-to-one, then f has an inverse function f^{-1} with domain $f(A)$. For the function f considered in Worked Exercise D40, it was possible to determine f^{-1} explicitly by solving the equation $y = f(x)$ to obtain x in terms of y. Unfortunately, it is often impossible to solve the equation $y = f(x)$ in this way.

Nevertheless, it may still be possible to prove that f has an inverse function f^{-1} by showing that f is one-to-one by some other method. For example in Unit A1 *Sets, functions and vectors* we showed that various functions f are one-to-one by proving algebraically that

$$\text{if } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

However, this algebraic method can only be used for fairly simple functions.

Another way of showing that f is one-to-one is by proving that f is *strictly increasing* or *strictly decreasing*. You met these concepts applied to real functions in Unit A4 and to sequences in Unit D2, where you also encountered the idea of a *monotonic* sequence. It is now helpful to use the term monotonic in the case of real functions also. The formal definitions are given in the box below, and are illustrated in the cases of strictly increasing and strictly decreasing functions in Figures 7 and 8 respectively.

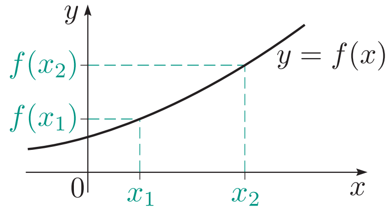


Figure 7 f is strictly increasing

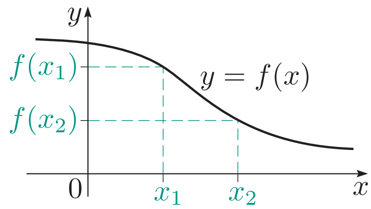


Figure 8 f is strictly decreasing

Definitions

A real function f defined on an interval I is

- **increasing** on I if
$$x_1 < x_2 \implies f(x_1) \leq f(x_2), \quad \text{for } x_1, x_2 \in I$$
- **strictly increasing** on I if
$$x_1 < x_2 \implies f(x_1) < f(x_2), \quad \text{for } x_1, x_2 \in I$$
- **decreasing** on I if
$$x_1 < x_2 \implies f(x_1) \geq f(x_2), \quad \text{for } x_1, x_2 \in I$$
- **strictly decreasing** on I if
$$x_1 < x_2 \implies f(x_1) > f(x_2), \quad \text{for } x_1, x_2 \in I$$
- **monotonic** on I if f is either increasing on I or decreasing on I
- **strictly monotonic** on I if f is either strictly increasing on I or strictly decreasing on I.

If the interval I in the definitions is the domain of f, then we omit 'on I' and just say, for example,

f is increasing.

The most powerful technique for proving that a function f is increasing or decreasing is to compute the derivative f' of f and examine the sign of $f'(x)$. We used this technique in Unit A4 and we will use it again in Book F *Analysis 2*, once we have laid a rigorous foundation for the idea of the derivative of a function. For the present, however, we consider only those functions which can be proved to be increasing or decreasing by manipulating inequalities using the rules from Unit D1 *Numbers*, rather than by using calculus.

For example, if $n \in \mathbb{N}$, then the function

$$f(x) = x^n \quad (x \in [0, \infty))$$

is strictly increasing; and if n is odd, then the function

$$f(x) = x^n \quad (x \in \mathbb{R})$$

is strictly increasing; see Figure 9(a).

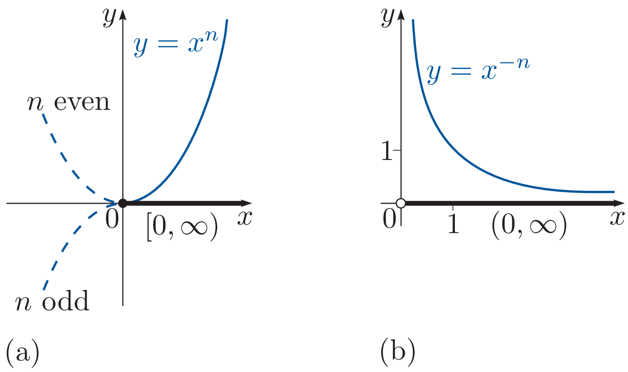


Figure 9 The graphs of (a) $y = x^n$ and (b) $y = x^{-n}$

Similarly, if $n \in \mathbb{N}$, then the function

$$f(x) = x^{-n} \quad (x \in (0, \infty))$$

is strictly decreasing; see Figure 9(b).

Any function that is strictly monotonic must be one-to-one since, if $x_1 < x_2$, then it is impossible to have $f(x_1) = f(x_2)$. Here is an example of how this property can be used.

Worked Exercise D41

Prove that the function

$$f(x) = x^5 + x - 1 \quad (x \in \mathbb{R})$$

is one-to-one.

Solution

If $x_1 < x_2$, then $x_1^5 < x_2^5$, so

$$x_1^5 + x_1 - 1 < x_2^5 + x_2 - 1; \quad \text{that is,} \quad f(x_1) < f(x_2).$$

Hence f is strictly increasing and thus one-to-one.

Exercise D57

Prove that the following functions are one-to-one.

(a) $f(x) = x^4 + 2x + 3 \quad (x \in [0, \infty))$

(b) $f(x) = \dfrac{1}{x} - x^2 \quad (x \in (0, \infty))$

Determining the image set

If the function $f : A \longrightarrow \mathbb{R}$ is strictly increasing or strictly decreasing, then f is one-to-one and so has an inverse function f^{-1} with domain $f(A)$. However, it is not always easy to determine the image set $f(A)$.

For example, consider the function

$$f(x) = x^5 + x - 1 \quad (x \in \mathbb{R}).$$

We saw in Worked Exercise D41 that f is one-to-one, so f has an inverse function with domain $f(\mathbb{R})$. Since f is strictly increasing, it seems likely that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is onto, so $f(\mathbb{R}) = \mathbb{R}$, and we have sketched the graph $y = f(x)$ in Figure 10 as though this is the case. But how can we *prove* that $f(\mathbb{R}) = \mathbb{R}$? To do this we want to show that, for each $y \in \mathbb{R}$, there is an x such that

$$f(x) = x^5 + x - 1 = y.$$

Unfortunately, we cannot find such an x by solving this equation algebraically to obtain x in terms of y. Could it be that the graph $y = x^5 + x - 1$ actually has some 'gaps' or 'jumps' in it? We would be very surprised if gaps do occur, but how can we prove that they do not?

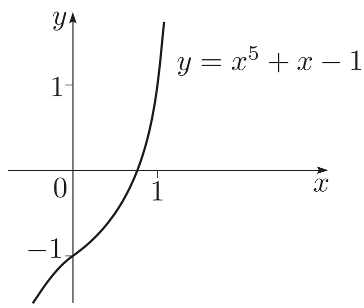


Figure 10 The graph of $y = x^5 + x - 1$

To answer this question, we need the concept of *continuity*, which we introduce in Section 2.

2 Continuous functions

In this section you will see what it means for a function f to be continuous at a point a. You will also meet several rules which enable you to combine continuous functions in various ways to form other continuous functions. Using these rules, together with a list of basic continuous functions, we can deduce that many functions are continuous at each point of their domains. For example, the function

$$x \longmapsto x\sin(1/x)$$

is continuous at each point of $\mathbb{R} - \{0\}$.

You will also meet rules which enable us to prove that certain hybrid functions are continuous. For example, we can show that the function

$$f(x) = \begin{cases} x\sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is continuous at 0. The graph of this function is shown in Figure 11.

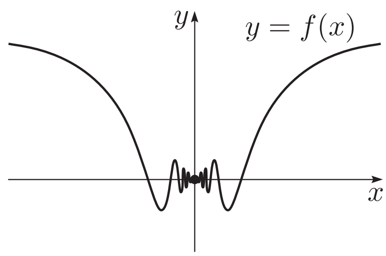


Figure 11 The graph of a continuous hybrid function

2.1 What is continuity?

At the end of Section 1 we asked whether the graph of the function

$$f(x) = x^5 + x - 1 \quad (x \in \mathbb{R})$$

has any gaps. Roughly speaking, can we draw the graph $y = x^5 + x - 1$ without lifting a pen from the paper? In this section we show that this graph cannot have any gaps because the function f is *continuous*.

Our first objective is to *define* the phrase

f is continuous at the point a.

To agree with our intuitive ideas, we wish to define this concept in such a way that the two functions shown in Figure 12 are continuous at the point a.

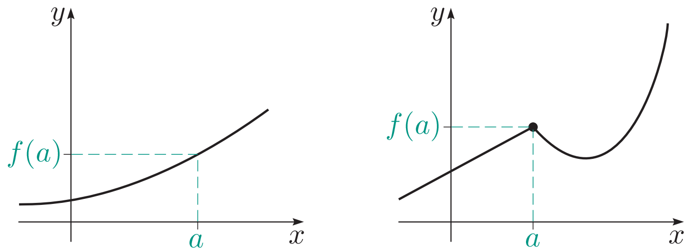


Figure 12 The graphs of two functions which are continuous at a

On the other hand, we wish to formulate our definition so that the two functions shown in Figure 13 are *not* continuous at the point a.

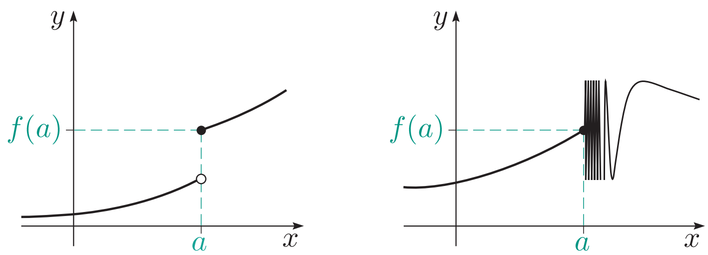


Figure 13 The graphs of two functions which are not continuous at a

For each of the graphs in Figure 12, as the values of x get closer and closer to a, the corresponding values of $f(x)$ get closer and closer to $f(a)$. This is not the case for the graphs in Figure 13. So our definition of continuity must say, in a precise way, that

if x tends to a, then $f(x)$ tends to $f(a)$.

There are several ways of making this idea precise. Here we adopt a definition that involves the *convergence of sequences*, since this enables us to use results about sequences that you have already met to prove that many common functions are continuous. (In Book F we give another definition of continuity which is convenient for dealing with more unusual functions.)

To motivate our definition of continuity we start with the following exercise. Here the sequences are denoted by (x_n) rather than (a_n) because, as you will see, they represent points on the x-axis.

Exercise D58

Let (x_n) be a sequence such that $\lim_{n\to\infty} x_n = 2$. Determine the limits of the following sequences.

(a) $(3x_n)$ (b) (x_n^2) (c) $(1/x_n)$

We now look again at these three limits, this time from a geometrical point of view.

We saw, in part (a) of Exercise D58, that

$$\text{if } x_n \to 2, \text{ then } 3x_n \to 6.$$

Using function notation with $f(x) = 3x$, we can express this statement in the equivalent form

$$\text{if } x_n \to 2, \text{ then } f(x_n) \to f(2).$$

This is illustrated in Figure 14.

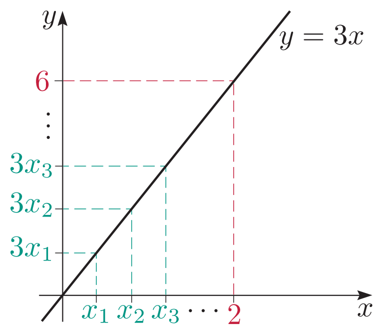


Figure 14 The graph of $y = 3x$

Next, we saw in part (b) of Exercise D58 that

$$\text{if } x_n \to 2, \text{ then } x_n^2 \to 4.$$

Using function notation with $f(x) = x^2$, we can express this statement in the equivalent form

$$\text{if } x_n \to 2, \text{ then } f(x_n) \to f(2).$$

This is illustrated in Figure 15.

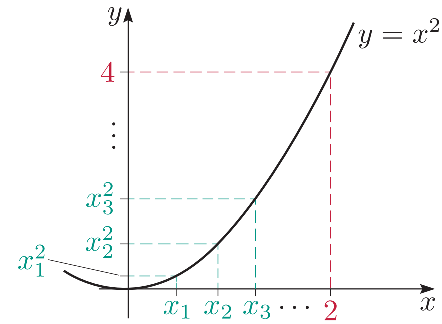


Figure 15 The graph of $y = x^2$

Finally, we saw, in part (c) of Exercise D58 that

$$\text{if } x_n \to 2, \text{ then } 1/x_n \to 1/2.$$

Using function notation with $f(x) = 1/x$, we can express this statement in the equivalent form

$$\text{if } x_n \to 2, \text{ then } f(x_n) \to f(2).$$

This is illustrated in Figure 16.

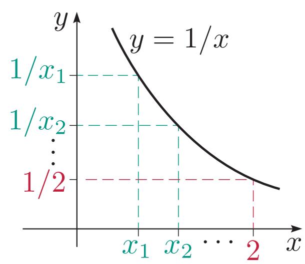


Figure 16 The graph of $y = 1/x$

So in each of these three cases, for any sequence (x_n) that has limit 2, we have shown that the sequence $f(x_n)$ has limit $f(2)$; that is,

$$\text{if } x_n \to 2, \text{ then } f(x_n) \to f(2).$$

In the above figures we have illustrated the situation in the case that the sequence (x_n) is increasing, but the conclusion holds no matter how (x_n) approaches 2.

In general, for a function $f : A \longrightarrow \mathbb{R}$, where A is a subset of $\mathbb{R}$ and $a \in A$, our formal definition of the continuity of f at a should encapsulate the

intuitive notion of continuity that, if (x_n) is any sequence such that x_n tends to a, then $f(x_n)$ tends to $f(a)$.

Our definition should also enable us to conclude that, if there is a sequence (x_n) such that x_n tends to a but $f(x_n)$ does *not* tend to $f(a)$, then f is *not continuous* at a. In such a situation we say that f is *discontinuous* at a. For instance, it may be that the graph of f near a has a jump at a.

These two different situations are illustrated in Figure 17.

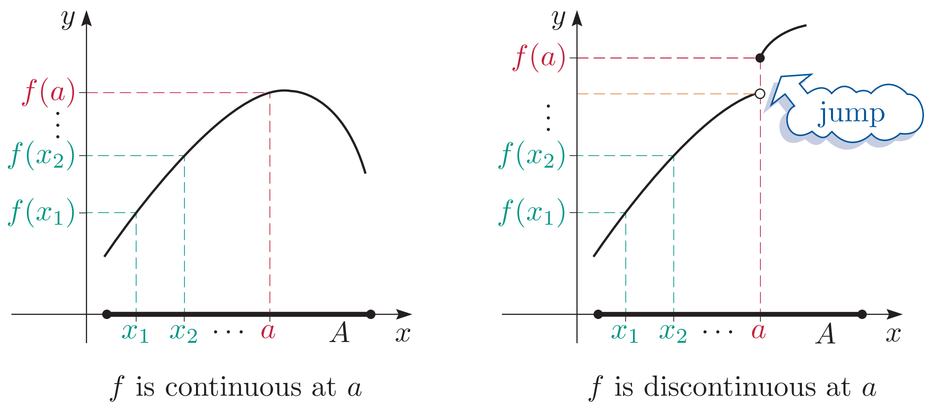


Figure 17 Continuity in terms of sequences

We now give our formal definition of continuity.

Definitions

A function $f : A \longrightarrow \mathbb{R}$ is **continuous** at a point $a \in A$ if for each sequence (x_n) in A such that $x_n \to a$, we have $f(x_n) \to f(a)$.

We say that f is **continuous** (on A) if f is continuous at each point $a \in A$.

If f is not continuous at a point a in A, then we say that f is **discontinuous** at a.

We can write the above condition for continuity more concisely as follows:

$$x_n \to a \implies f(x_n) \to f(a), \text{ where } (x_n) \text{ lies in } A.$$

The next two worked exercises illustrate how we can use the definition of continuity to show that a function is continuous at a given point or discontinuous at a given point.

Worked Exercise D42

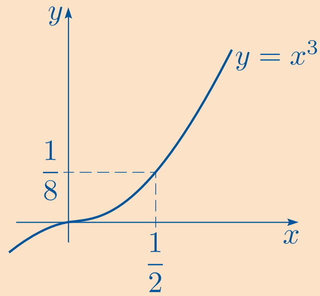
Prove that the function

$$f(x) = x^3 \quad (x \in \mathbb{R})$$

is continuous at the point $1/2$.

Solution

The graph of f is shown below. It appears from the graph that f is continuous at $1/2$ but we must now prove that this is true.



Let (x_n) be *any* sequence in $\mathbb{R}$ that converges to $1/2$; that is, $x_n \to 1/2$ as $n \to \infty$. Then, by the Product Rule for sequences, it follows that

$$f(x_n) = x_n^3 \to (1/2)^3 = 1/8 \text{ as } n \to \infty.$$

Since $f(1/2) = 1/8$, we have

$$f(x_n) \to f(1/2) \text{ as } n \to \infty.$$

It follows that f is continuous at $1/2$, as required.

Worked Exercise D43

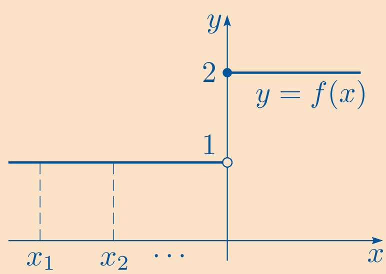
Prove that the function

$$f(x) = \begin{cases} 1, & x < 0, \\ 2, & x \geq 0, \end{cases}$$

is discontinuous at 0.

Solution

In order to prove that a function is discontinuous at a particular point, we just need to find *one* sequence for which our definition of continuity at the relevant point does *not* hold. We begin by looking at the graph of f to find a suitable sequence.



Let (x_n) be any sequence in $\mathbb{R}$ that converges to 0 from the *right* as $n \to \infty$. By looking at the graph of f, it is clear that for such a sequence $f(x_n) \to 2 = f(0)$. This won't help us to show that f is discontinuous at 0. However, if we let (x_n) be any sequence in $\mathbb{R}$ that converges to 0 from the *left*, the situation is very different. By looking at the graph of f, it is clear that, for such a sequence, $f(x_n) \to 1 \neq f(0)$. We choose $x_n = -1/n$ as this is a simple sequence of this type. We now make this reasoning precise.

We first note that $f(0) = 2$. We now choose

$$x_n = -1/n, \quad n = 1, 2, \ldots.$$

Then $x_n \to 0$ and, since $x_n < 0$, we have $f(x_n) = 1$ for $n = 1, 2, \ldots$, so

$$f(x_n) \to 1 \neq f(0) \text{ as } n \to \infty.$$

Hence f is discontinuous at 0.

Worked Exercises D42 and D43 illustrate the following general strategy.

Strategy D14

- To prove that a function $f : A \longrightarrow \mathbb{R}$ is *continuous* at the point $a \in A$:

  show that, if (x_n) is *any* sequence in A such that $x_n \to a$, then $f(x_n) \to f(a)$.

- To prove that a function $f : A \longrightarrow \mathbb{R}$ is *discontinuous* at the point $a \in A$:

  find *one* sequence (x_n) in A such that $x_n \to a$ but $f(x_n) \not\to f(a)$.

Remarks

1. The symbol $\not\to$ is read as 'does not tend to'.
2. When proving discontinuity at a point a it is often possible to choose the sequence to be $(a - 1/n)$ or $(a + 1/n)$.

We now illustrate the use of this strategy.

Worked Exercise D44

(a) Determine whether the function

$$f(x) = 1/x \quad (x \in \mathbb{R} - \{0\})$$

is continuous.

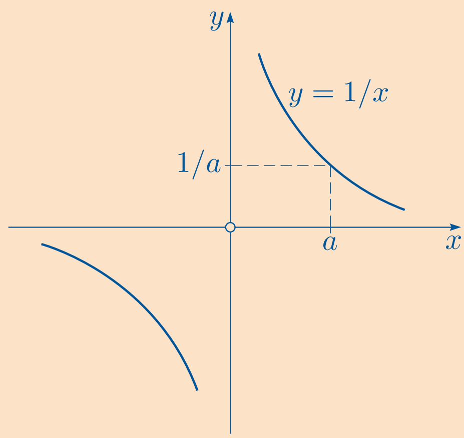
(b) Determine whether the function

$$f(x) = \begin{cases} 1/x, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is continuous at 0.

Solution

(a) Recall that a function is continuous if it is continuous at each point in its domain. The graph suggests that f is continuous everywhere it is defined.



We now prove that this is the case using Strategy D14.

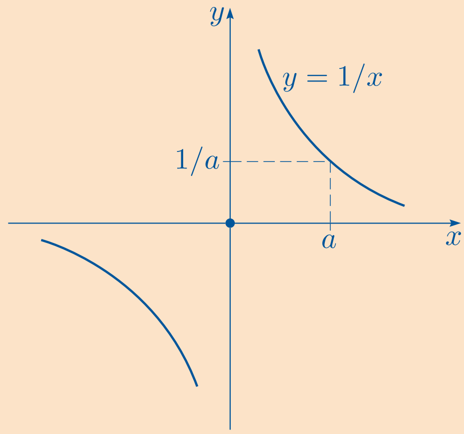
The function f has domain $\mathbb{R} - \{0\}$.

If $a \neq 0$ and (x_n) is a sequence in $\mathbb{R} - \{0\}$ with $x_n \to a$, then, by the Quotient Rule for sequences,

$$f(x_n) = 1/x_n \to 1/a = f(a).$$

It follows that f is continuous at every $a \neq 0$ and hence at every point in its domain.

(b) This function is the same as in part (a) apart from at 0 which is now in the domain. The graph of f has a jump at 0, so it certainly looks as though f is discontinuous at 0.



We now prove this using Strategy D14.

We first note that $f(0) = 0$.

We now choose

$$x_n = 1/n, \quad n = 1, 2, \dots .$$

Then $x_n \to 0$ but

$$f(x_n) = f(1/n) = n \to \infty,$$

so $f(x_n) \nrightarrow f(0) = 0$. It follows that the function f is discontinuous at 0.

Exercise D59

Determine whether the following functions are continuous at the points given. (Remember that $\lfloor x \rfloor$ denotes the integer part of x.)

(a) $f(x) = x^3 - 2x^2$, at the point $a = 2$

(b) $f(x) = \lfloor x \rfloor$, at the point $a = 1$

Exercise D60

Prove that the following functions are continuous (that is, continuous at every point $a \in \mathbb{R}$).

(a) $f(x) = 1$

(b) $f(x) = x$

We conclude this subsection with two worked exercises in which we prove the continuity of two important functions: the modulus function and the square root function.

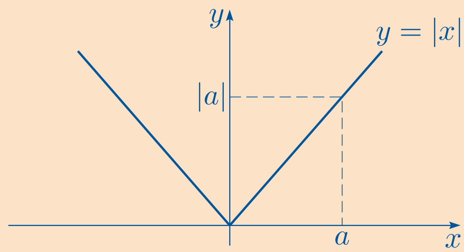
Worked Exercise D45

Determine whether the following function is continuous.

$$f(x) = |x| \quad (x \in \mathbb{R})$$

Solution

It certainly seems from the graph that f is continuous on its domain $\mathbb{R}$.



We guess that f is continuous on its domain $\mathbb{R}$. Let $a \in \mathbb{R}$, and let (x_n) be any sequence in $\mathbb{R}$ with $x_n \to a$ as $n \to \infty$. We want to prove that

$$x_n \to a \implies |x_n| \to |a|,$$

or, in other words, that

$$(x_n - a) \text{ is a null sequence} \implies (|x_n| - |a|) \text{ is a null sequence.} \quad (*)$$

To prove statement $(*)$, we use the backwards form of the Triangle Inequality that you met in Unit D1. This says that for any $a, b \in \mathbb{R}$, we have $|a - b| \geq \big||a| - |b|\big|$.

Now it follows from the backwards form of the Triangle Inequality that

$$|x_n - a| \geq \big||x_n| - |a|\big|, \quad \text{for } n = 1, 2, \ldots$$

We now see that the sequence $(|x_n| - |a|)$ is 'squeezed' between the null sequence $(x_n - a)$ and the constant null sequence (0).

Thus, by the Squeeze Rule for sequences, statement $(*)$ holds.

Hence f is continuous.

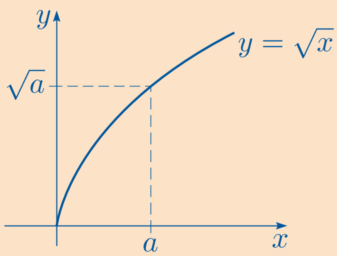
Worked Exercise D46

Determine whether the following function is continuous.

$$f(x) = \sqrt{x} \quad (x \in [0, \infty))$$

Solution

It certainly seems from the graph that f is continuous on its domain, $[0, \infty)$.



We guess that f is continuous on its domain $[0, \infty)$.

Let $a \in [0, \infty)$, and let (x_n) be any sequence in $[0, \infty)$ with $x_n \to a$ as $n \to \infty$. We want to prove that

$$x_n \to a \implies \sqrt{x_n} \to \sqrt{a},$$

or, in other words, that

$$(x_n - a) \text{ is a null sequence} \implies \left(\sqrt{x_n} - \sqrt{a}\right) \text{ is a null sequence.} \ (*)$$

Now, since $(x_n - a)$ is null, the sequence $(|x_n - a|)$ is null and hence, by the Power Rule for sequences, the sequence $\left(\sqrt{|x_n - a|}\right)$ is also null.

Next we use an inequality proved in Subsection 3.2 of Unit D1, namely that if $a \geq 0$ and $b \geq 0$, then $\sqrt{|a - b|} \geq \left|\sqrt{a} - \sqrt{b}\right|$.

Moreover, since $x_n \geq 0$ and $a \geq 0$, it follows that

$$\sqrt{|x_n - a|} \geq \left|\sqrt{x_n} - \sqrt{a}\right|, \quad \text{for } n = 1, 2, \ldots.$$

We now see that the sequence $\left(\sqrt{x_n} - \sqrt{a}\right)$ is 'squeezed' between the null sequence $\left(\sqrt{|x_n - a|}\right)$ and the constant null sequence (0).

Thus, by the Squeeze Rule for sequences, statement $(*)$ holds.

Hence f is continuous.

2.2 Rules for continuous functions

In the previous subsection you saw how to use the definition to check whether a given function is continuous at a point. You will now meet a number of rules that can be used to show that many functions are continuous without the need to go back to the definition. The first set of rules is called the Combination Rules.

Theorem D40 Combination Rules for continuous functions

If f and g are continuous at a, then so are

Sum Rule	$f+g$
Multiple Rule	λf, for $\lambda \in \mathbb{R}$
Product Rule	fg
Quotient Rule	f/g, provided that $g(a) \neq 0$.

Proof The proofs of these rules are similar and depend on the corresponding results for sequences. We prove only the Sum Rule.

Suppose that f and g are continuous at a. We want to deduce that $f+g$ is continuous at a.

Let the domain of f be A and the domain of g be B. Then the domain of $f+g$ is $A \cap B$ and this set contains a.

Thus, since

$$(f+g)(x) = f(x) + g(x), \quad \text{for } x \in A \cap B,$$

we have to show that

for each sequence (x_n) in $A \cap B$ such that $x_n \to a$, we have

$$f(x_n) + g(x_n) \to f(a) + g(a). \tag{1}$$

We know that (x_n) lies in A and in B, and that both functions f and g are continuous at a. Hence

$$f(x_n) \to f(a) \quad \text{and} \quad g(x_n) \to g(a),$$

so statement (1) follows by the Sum Rule for sequences. ■

By using the Combination Rules we can show, for example, that the function $f(x) = 1 - 2x$ is continuous (on $\mathbb{R}$) since

$x \longmapsto x$ is continuous (by Exercise D60(b))

and so

$x \longmapsto -2x$ is continuous by the Product Rule.

Also

$x \longmapsto 1$ is continuous (by Exercise D60(a))

and so

$x \longmapsto 1 - 2x$ is continuous by the Sum Rule.

Indeed any polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n, x \in \mathbb{R}$, is continuous at all points of $\mathbb{R}$ since we can build up the expression for p by successive applications of the Combination Rules.

Moreover, it then follows from the Quotient Rule that any rational function $r(x) = p(x)/q(x)$, where p and q are polynomials, is continuous at all points of its domain, that is, all of $\mathbb{R}$ except for points where $q(x) = 0$. We have thus established the following result.

Theorem D41

The following functions are continuous:

- any polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$ (on its domain $\mathbb{R}$)
- any rational function $r(x) = p(x)/q(x)$, where p and q are polynomials (on its domain $\mathbb{R} - \{x : q(x) = 0\}$).

The Combination Rules are natural analogues of the corresponding results for sequences. However, we can combine functions in more ways than we can combine sequences: for example, we can *compose* functions f and g to obtain the function $g \circ f$. Composing functions also enables us to obtain 'new continuous functions from old', as the next rule shows.

Theorem D42 Composition Rule for continuous functions

If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

Proof Suppose that f is continuous at a and g is continuous at $f(a)$. We want to deduce that $g \circ f$ is continuous at a.

If f has domain A and g has domain B, then the domain of $g \circ f$ is

$$C = \{x \in A : f(x) \in B\}$$

and this set contains a.

Thus we have to show that

for each sequence (x_n) in C such that $x_n \to a$, we have
$$g(f(x_n)) \to g(f(a)). \tag{2}$$

We know that (x_n) lies in A and that f is continuous at a, so $f(x_n) \to f(a)$. Moreover, because (x_n) lies in C, we also know that $(f(x_n))$ lies in B, and since g is continuous at $f(a)$, it follows that $g(f(x_n)) \to g(f(a))$. Hence statement (2) is true. ■

Worked Exercise D47

Determine whether the following function is continuous.

$$h(x) = \sqrt{x^2+1} \quad (x \in \mathbb{R})$$

Solution

We guess that h is continuous on $\mathbb{R}$.

If we let $f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$, so that $g(f(x)) = \sqrt{x^2+1}$, we see that we can express h as a composite function $h = g \circ f$.

Now, f is continuous (on $\mathbb{R}$), since it is a polynomial, and all its values are positive. Also, g is continuous on $[0, \infty)$, by Worked Exercise D46. It then follows from the Composition Rule that $h = g \circ f$ is continuous.

Exercise D61

Determine whether the following function is continuous.

$$f(x) = |x^5| \quad (x \in [0, \infty))$$

Exercise D62

Prove that the following function is continuous, stating each rule or fact that you use.

$$f(x) = \sqrt{x^2+2x+2} - \frac{3x}{x^4+4} \quad (x \in \mathbb{R})$$

Next, just as we had a Squeeze Rule for convergent sequences, we show that there is a corresponding Squeeze Rule for continuous functions. The graphs of the functions in the next theorem are illustrated in Figure 18.

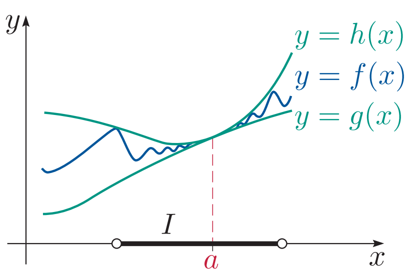


Figure 18 The functions in the Squeeze Rule

Theorem D43 Squeeze Rule for continuous functions

Let f, g and h be defined on an open interval I and let $a \in I$. If

1. $g(x) \le f(x) \le h(x)$, for $x \in I$,
2. $g(a) = f(a) = h(a)$, and
3. g and h are continuous at a,

then f is also continuous at a.

Proof Suppose that f, g and h satisfy the conditions of the theorem. We want to prove that f is continuous at a.

Thus we have to show that

for each sequence (x_n) in the domain of f such that $x_n \to a$, we have $f(x_n) \to f(a)$. (3)

Since $x_n \to a$ and I is an open interval, there is an integer N such that

$$x_n \in I, \quad \text{for } n > N.$$

Hence, by condition 1,

$$g(x_n) \le f(x_n) \le h(x_n), \quad \text{for } n > N.$$

By conditions 2 and 3,

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} h(x_n) = f(a),$$

so statement (3) follows, by the Squeeze Rule for sequences. ■

In the following worked exercise, the only property of the sine function that we need is that $|\sin x| \le 1$, for $x \in \mathbb{R}$. (We will investigate whether the trigonometric functions sin, cos and tan are continuous in the next subsection.)

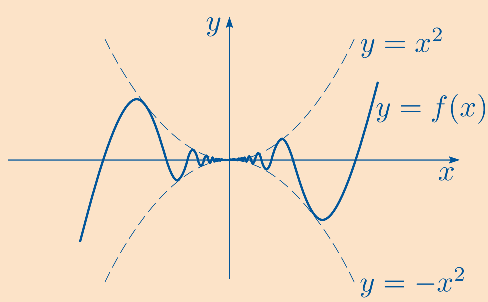
Worked Exercise D48

Determine whether the following function is continuous at 0.

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Solution

The graph of f is shown below.



This suggests that we can find functions g and h that squeeze f near 0, and then use the Squeeze Rule.

We use the Squeeze Rule.

We know that

$$-1 \leq \sin(1/x) \leq 1, \quad \text{for } x \neq 0.$$

Since $x^2 \geq 0$, it follows that

$$-x^2 \leq x^2 \sin(1/x) \leq x^2, \quad \text{for } x \neq 0.$$

Thus, since $f(0) = 0$, we have

$$-x^2 \leq f(x) \leq x^2, \quad \text{for } x \in \mathbb{R}.$$

If we now take $I = \mathbb{R}$, with

$$g(x) = -x^2 \quad \text{and} \quad h(x) = x^2,$$

then

$$g(x) \leq f(x) \leq h(x), \quad \text{for } x \in I,$$

so condition 1 of the Squeeze Rule is satisfied.

Next, $f(0) = g(0) = h(0) = 0$, so condition 2 of the Squeeze Rule is satisfied.

Finally, the functions g and h are polynomials, and so in particular they are continuous at 0. Thus condition 3 of the Squeeze Rule is satisfied.

It then follows from the Squeeze Rule that f is continuous at 0, as required.

Exercise D63

Determine whether the following functions are continuous at 0.

(a) $f(x) = \begin{cases} x^2 \cos(1/x^2), & x \neq 0, \\ 0, & x = 0. \end{cases}$

(b) $f(x) = \begin{cases} \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$

Hint: Use the fact that $\sin\left(2n + \frac{1}{2}\right)\pi = 1$, for $n \in \mathbb{Z}$.

We now describe another rule for proving that a function is continuous at a point. Consider the hybrid function

$$f(x) = \begin{cases} 1 - x, & x < 0, \\ 0, & 0 \leq x \leq 1, \\ 3x - 3, & x > 1. \end{cases} \tag{4}$$

The domain of f is the whole of $\mathbb{R}$ and the graph of f is shown in Figure 19.

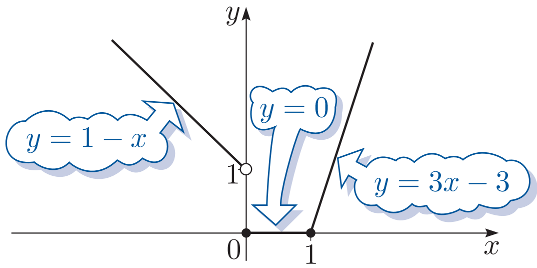


Figure 19 The graph of the hybrid function $y = f(x)$

From the graph, it appears that

1. f is discontinuous at $a = 0$
2. f is continuous at all other values of a.

We can prove that f is discontinuous at 0 by using Strategy D14. We need to find *one* sequence (x_n) such that

$$x_n \to 0 \quad \text{but} \quad f(x_n) \nrightarrow f(0).$$

Since $f(0)$ is defined using the rule for $[0, 1]$, we choose a simple sequence (x_n) which tends to 0 from the left; we can choose

$$x_n = -\frac{1}{n}, \quad n = 1, 2, \ldots.$$

This is illustrated in Figure 20.

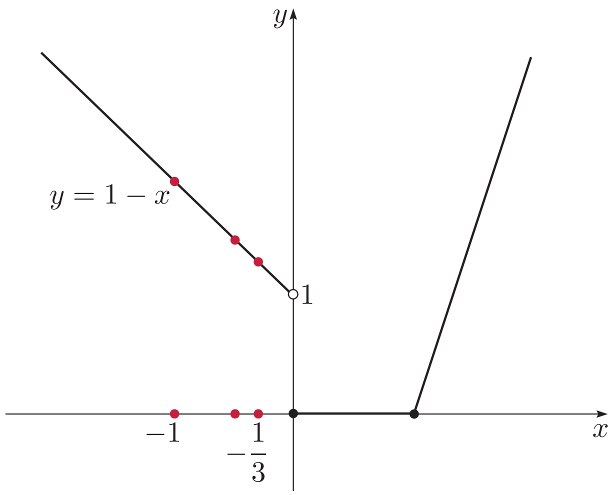


Figure 20 A sequence whose images under f do not tend to $f(0)$

The rule for $f(x)$ which applies for $x < 0$ is $1 - x$, so

$$f(x_n) = f\left(-\frac{1}{n}\right) = 1 - \left(-\frac{1}{n}\right) = 1 + \frac{1}{n}, \quad \text{for } n = 1, 2, \ldots,$$

and we also have $f(0) = 0$. Hence

$$x_n \to 0 \quad \text{but} \quad f(x_n) \to 1 \neq f(0).$$

Thus f is discontinuous at 0.

But how can we prove that f *is continuous* at $a = 1$, as the graph suggests? If we define

$$g(x) = 0 \quad (x \in \mathbb{R}) \quad \text{and} \quad h(x) = 3x - 3 \quad (x \in \mathbb{R}),$$

then near the point 1, the graph of f consists of part of the graph $y = g(x)$ to the left of 1, glued to part of the graph $y = h(x)$ to the right of 1.

This idea is the basis of the Glue Rule which is illustrated in Figure 21.

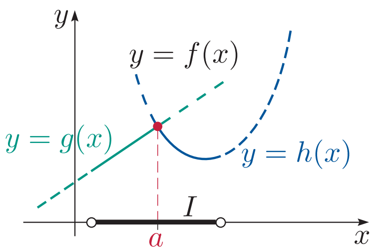


Figure 21 The Glue Rule

Theorem D44 Glue Rule for continuous functions

Let f be defined on an open interval I and let $a \in I$.
If there are functions g and h such that

1. $f(x) = g(x)$, for $x \in I$, $x < a$,
   $f(x) = h(x)$, for $x \in I$, $x > a$,
2. $f(a) = g(a) = h(a)$, and
3. g and h are continuous at a,

then f is also continuous at a.

Proof Suppose that f, g and h satisfy the conditions of the theorem. We want to prove that

for each sequence (x_n) in the domain of f such that $x_n \to a$, we have $f(x_n) \to f(a)$. (5)

Since $x_n \to a$ and I is an open interval, there is an integer N such that

$$x_n \in I, \quad \text{for } n \geq N.$$

Then $(x_n)_N^\infty$ (that is, the sequence $x_N, x_{N+1}, \ldots$) consists of two subsequences (x_{m_k}) and (x_{n_k}) defined by the conditions

$$x_{m_k} < a, \quad \text{for } k = 1, 2, \ldots, \quad \text{and} \quad x_{n_k} \geq a, \quad \text{for } k = 1, 2, \ldots.$$

By conditions 1 and 3, we have

$$g(x_{m_k}) \to g(a) \text{ as } k \to \infty \quad \text{and} \quad h(x_{n_k}) \to h(a) \text{ as } k \to \infty.$$

Hence, by conditions 1 and 2, we have

$$f(x_{m_k}) \to f(a) \text{ as } k \to \infty \quad \text{and} \quad f(x_{n_k}) \to f(a) \text{ as } k \to \infty.$$

We have shown that the sequence $(f(x_n))$ consists of two subsequences with the same limit, $f(a)$. Recall that, by Theorem D21 in Unit D2, it follows that the whole sequence converges to this limit.

Since both subsequences of $(f(x_n))$ tend to $f(a)$, statement (5) follows. ■

Note that the Glue Rule does not require the functions g and h to be defined on the whole of I, though this is often the case.

We now apply the Glue Rule to the function that we were looking at earlier.

Worked Exercise D49

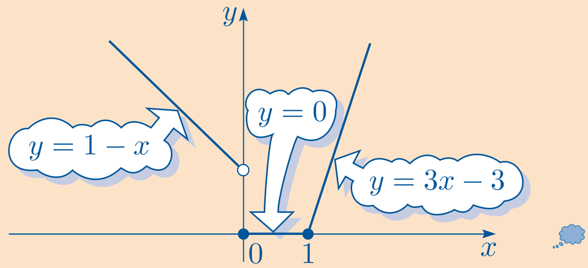
Use the Glue Rule to prove that the function

$$f(x) = \begin{cases} 1-x, & x<0, \\ 0, & 0 \le x \le 1, \\ 3x-3, & x>1, \end{cases}$$

is continuous at 1.

Solution

The graph of f is shown again below.

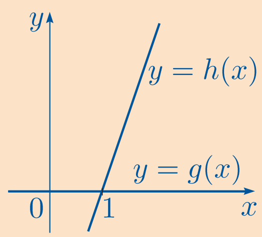


$y = f(x)$

Let I be the open interval $(0,\infty)$ and define the functions

$$g(x) = 0 \quad (x \in \mathbb{R}) \quad \text{and} \quad h(x) = 3x-3 \quad (x \in \mathbb{R}).$$

The graphs of g and h are shown below.



We chose $I = (0,\infty)$ because on this interval the rule of f is given by the rule of g to the left of 1 and by the rule of h to the right of 1. We could have chosen any smaller open interval that contains the point 1.

Then f is defined on I and $1 \in I$. Also,

$$f(x) = g(x), \quad \text{for } x \in (0,1),$$
$$f(x) = h(x), \quad \text{for } x \in (1,\infty),$$

so condition 1 of the Glue Rule holds with $a = 1$.

Moreover, $f(1) = g(1) = h(1) = 0$, so condition 2 holds.

Finally, g and h are both polynomials and are therefore both continuous at 1, so condition 3 holds.

Hence f is continuous at 1, by the Glue Rule.

Exercise D64

Prove that the following function is continuous at 1.

$$f(x) = \begin{cases} x^3 - 3x + 5, & x < 1, \\ \dfrac{2x+1}{3x-2}, & x \geq 1. \end{cases}$$

There are two further straightforward situations in which we can obtain 'new continuous functions from old' and we illustrate these by examples. When such situations arise, we normally use the results illustrated here without explicitly referring to them.

1. Consider the function

$$f(x) = \begin{cases} 1 - x, & x < 0, \\ 0, & 0 \leq x \leq 1, \\ 3x - 3, & x > 1, \end{cases}$$

which we studied in Worked Exercise D49. There we used the Glue Rule to show that f is continuous at 1. We also showed earlier that f is discontinuous at 0. It seems evident that this function f is continuous at all other points in $\mathbb{R}$. For example, the continuity of f at -1 depends only on the values taken by the function f *near* the point -1, and these values are the same as those of the function

$$g(x) = 1 - x \quad (x \in \mathbb{R}),$$

which is continuous since it is a polynomial. Since g is continuous at -1, we deduce that f is also continuous at -1. A similar argument can be used to show that f is continuous at all points $a \in \mathbb{R} - \{0, 1\}$. Since we have already shown by the Glue Rule that f is continuous at 1, we conclude that f is continuous on $\mathbb{R} - \{0\}$, as expected.

Notice that the above argument works because continuity at a point is a *local property*; that is, it depends only on the behaviour of the function near that point.

2. Consider the function

$$f(x) = x^2 \quad (x \in [-1, 1]).$$

The domain of this function is $[-1, 1]$, and it certainly appears from Figure 22 that f is continuous at each point of $[-1, 1]$. After all, the function

$$g(x) = x^2 \quad (x \in \mathbb{R})$$

is a basic continuous function (since it is a polynomial).

Recall from Unit A1 that if the domain A of a function h is a subset of the domain of a function k and $h(x) = k(x)$ for all $x \in A$, then h is

called the **restriction** of k to A. Thus here f is the restriction of g to the set $[-1, 1]$.

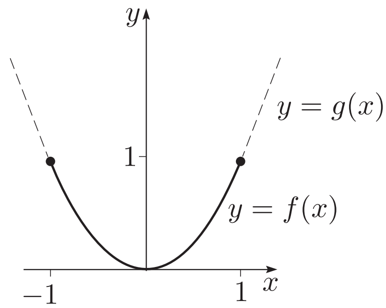


Figure 22 The graph of the restriction of $y = x^2$ to $[-1, 1]$

If we again note that continuity is a local property, it is easy to see from the definition of continuity that if a function f is the restriction of another function g, and g is continuous, then f is also continuous. We can sum up what we have shown as follows:

The restriction of a continuous function is continuous.

2.3 Trigonometric functions and the exponential function

We now prove that the trigonometric functions sin, cos and tan and the exponential function are continuous at all points of $\mathbb{R}$ where they are defined.

Trigonometric functions

We start with a basic inequality for the sine function. This is illustrated in Figure 23.

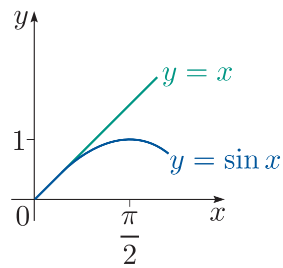


Figure 23 The graphs of $y = \sin x$ and $y = x$

Theorem D45 Sine Inequality

$$\sin x \le x, \quad \text{for } 0 \le x \le \frac{\pi}{2}.$$

Proof If $x = 0$, then $\sin 0 = 0$, so there is equality.

Suppose next that $0 < x \le \pi/2$, and consider the diagram in Figure 24, which represents a quarter circle, centred at the origin, with radius 1.

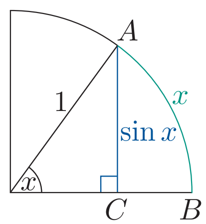


Figure 24 An arc of a circle of length x

Since the circle has radius 1, the arc AB has length x equal to the angle x (measured in radians) and the perpendicular AC has length $\sin x$. Hence

$$\sin x < x, \quad \text{for } 0 < x \leq \frac{\pi}{2}.$$

Combining this inequality with the fact that $\sin 0 = 0$ gives the result. ■

We can now deduce the crucial inequality for proving the continuity of the sine function. This is illustrated in Figure 25.

Corollary D46

$$|\sin x| \leq |x|, \quad \text{for } x \in \mathbb{R}.$$

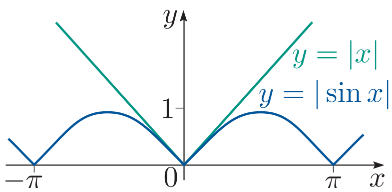


Figure 25 The graphs of $y = |\sin x|$ and $y = |x|$

Proof The Sine Inequality shows that this inequality holds for $0 \leq x \leq \pi/2$. For $x > \pi/2$, we have

$$|\sin x| \leq 1 < \frac{\pi}{2} < x = |x|,$$

so the inequality is also true in this case.

Finally, the inequality holds for $x < 0$, since

$$|\sin(-x)| = |\sin x| \quad \text{and} \quad |-x| = |x|.$$ ■

This is the key tool that we need to prove the continuity of the trigonometric functions.

Theorem D47

The trigonometric functions sine, cosine and tangent are continuous.

Proof To prove that the sine function is continuous at $a \in \mathbb{R}$, we want to show that

for each sequence (x_n) in $\mathbb{R}$ such that $x_n \to a$, we have
$$\sin x_n \to \sin a. \tag{6}$$

To do this, we use the trigonometric identity

$$\sin x - \sin a = 2\cos\left(\tfrac{1}{2}(x+a)\right)\sin\left(\tfrac{1}{2}(x-a)\right).$$

This identity can be obtained by writing

$$x = \tfrac{1}{2}(x+a) + \tfrac{1}{2}(x-a)$$

and

$$a = \tfrac{1}{2}(x+a) - \tfrac{1}{2}(x-a),$$

and then using the formulas for $\sin(A+B)$ and $\sin(A-B)$ which you can find in the module Handbook.

We obtain

$$\begin{aligned}|\sin x_n - \sin a| &= \left|2\cos\left(\tfrac{1}{2}(x_n + a)\right)\sin\left(\tfrac{1}{2}(x_n - a)\right)\right| \\ &\le 2\left|\sin\left(\tfrac{1}{2}(x_n - a)\right)\right| \quad \text{(since } |\cos x| \le 1\text{)} \\ &\le 2\left|\tfrac{1}{2}(x_n - a)\right| \quad \text{(by Corollary D46)} \\ &= |x_n - a|.\end{aligned}$$

Thus, if $(x_n - a)$ is null, then $(\sin x_n - \sin a)$ is null, by the Squeeze Rule for sequences, so statement (6) holds.

The continuity of the cosine and tangent functions now follows from the identities

$$\cos x = \sin\left(x + \tfrac{1}{2}\pi\right) \quad \text{and} \quad \tan x = \frac{\sin x}{\cos x},$$

using the Composition Rule and the Quotient Rule. ■

Exercise D65

Prove that the following function is continuous (on $\mathbb{R}$), stating each rule or fact about continuity that you use.

$$f(x) = x^2 + 1 + 3\sin\left(\sqrt{x^2+1}\right).$$

The exponential function

We start with two fundamental inequalities for the exponential function. These are illustrated in Figure 26.

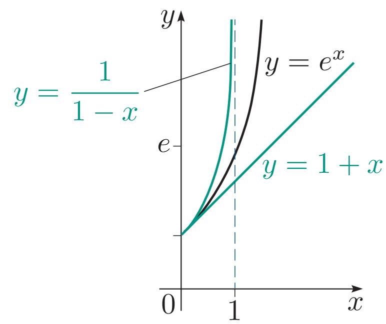


Figure 26 The graphs of $y = e^x$, $y = 1/(1-x)$ and $y = 1 + x$

Theorem D48 Exponential Inequalities

(a) $e^x \ge 1 + x$, for $x \ge 0$

(b) $e^x \le \dfrac{1}{1-x}$, for $0 \le x < 1$.

Proof We prove both inequalities using the exponential series which you met in Unit D3 *Series*:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad \text{for } x \ge 0.$$

(a) For $x \ge 0$, we have $x^2/2! \ge 0$, $x^3/3! \ge 0$, and so on. Hence

$$e^x \ge 1 + x, \quad \text{for } x \ge 0.$$

(b) For $x \ge 0$, we also have $x^2/2! \le x^2$, $x^3/3! \le x^3$, and so on. Hence

$$e^x \le 1 + x + x^2 + x^3 + \cdots.$$

The series on the right is a geometric series, which is convergent with sum $1/(1-x)$, for $0 \le x < 1$. Hence

$$e^x \le \frac{1}{1-x}, \quad \text{for } 0 \le x < 1.$$

■

We can now deduce the following inequalities which we use in proving the continuity of the exponential function. These are illustrated in Figure 27.

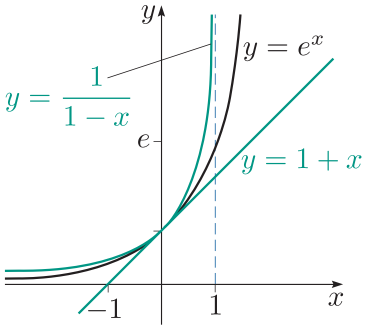


Figure 27 The graphs of the functions in Corollary D49

Corollary D49

$$1 + x \le e^x \le \frac{1}{1-x}, \quad \text{for } |x| < 1.$$

Proof The Exponential Inequalities show that these inequalities both hold for $0 \le x < 1$. For $-1 < x < 0$, we have $0 < -x < 1$, so

$$1 + (-x) \le e^{-x} \le \frac{1}{1-(-x)}.$$

Taking reciprocals and reversing these inequalities (which is possible since all three expressions are positive for $-1 < x < 0$), we obtain

$$1 + x \le e^x \le \frac{1}{1-x}, \quad \text{for } -1 < x < 0.$$

Hence

$$1 + x \le e^x \le \frac{1}{1-x}, \quad \text{for } |x| < 1.$$

■

We can now prove the continuity of the exponential function.

Theorem D50

The exponential function is continuous.

Proof To prove that the exponential function is continuous at $a \in \mathbb{R}$, we want to show that

for each sequence (x_n) in $\mathbb{R}$ such that $x_n \to a$, we have
$$e^{x_n} \to e^a. \tag{7}$$

Now, if $(x_n - a)$ is null, then there is a positive integer N such that $|x_n - a| < 1$, for $n > N$. Applying Corollary D49, with $x_n - a$ instead of x, we obtain

$$1 + (x_n - a) \le e^{x_n - a} \le \frac{1}{1-(x_n - a)}, \quad \text{for } n > N.$$

Thus $e^{x_n - a} \to 1$, by the Squeeze Rule for sequences. Hence $e^{x_n} = e^a e^{x_n - a} \to e^a$, so statement (7) holds. ■

Exercise D66

Prove that the following function is continuous, stating each rule or fact about continuity that you use.

$$f(x) = x^5 - 5x^2 + 7e^{-x^2}$$

We end this subsection with a reminder of the various approaches that you have met for investigating the continuity of a function $f : A \longrightarrow \mathbb{R}$ at $a \in A$. Recall that you should first guess whether f is continuous or discontinuous at a, then check whether your guess is correct (a sketch of the graph may help you make your guess). You can check your guess using Strategy D14. You have also seen that, in many cases, it is possible to show that f is continuous at a by applying rules such as the Combination Rules, the Composition Rule, the Squeeze Rule and the Glue Rule to functions which you already know to be continuous. We have proved that a number of familiar functions are continuous and we now collect these together in the following result.

Theorem D51 Basic continuous functions

The following functions are continuous:

- polynomials and rational functions
- $f(x) = |x|$
- $f(x) = \sqrt{x}$
- the trigonometric functions sine, cosine and tangent
- the exponential function.

3 Properties of continuous functions

In this section you will meet some of the fundamental properties of continuous functions, and see that these properties hold for continuous functions defined on **bounded closed intervals**; that is, intervals of the form $[a, b]$. You will also see some applications of these properties, in particular to locating the zeros of a continuous function.

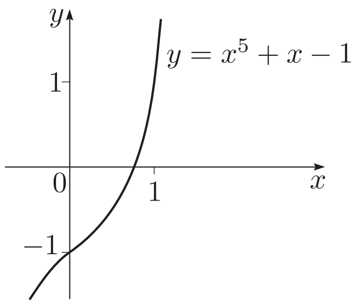


Figure 28 The graph of $y = x^5 + x - 1$

3.1 Intermediate Value Theorem

At the end of Section 1 we considered the function

$$f(x) = x^5 + x - 1 \quad (x \in \mathbb{R})$$

and we pointed out that it is not easy to prove that $f(\mathbb{R}) = \mathbb{R}$.

For example, is there a value of x such that $f(x) = 0$? In other words, is there a solution of the equation

$$x^5 + x - 1 = 0?$$

The shape of the graph of $y = x^5 + x - 1$ shown in Figure 28 certainly suggests that such a number x exists. Since $f(0) = -1$ and $f(1) = 1$, we expect there to be some number x in the interval $(0, 1)$ such that $f(x) = 0$. However, we do not have a formula for solving the above equation to find x.

The key to showing that such a number x exists lies in the fact that f is a continuous function, so there cannot be any gaps in its graph; this is the essence of the *Intermediate Value Theorem*. We prove this result at the end of this subsection but first we show how it can be used.

Theorem D52 Intermediate Value Theorem

Let f be a function continuous on $[a, b]$ such that $f(a) \neq f(b)$, and let k be any number lying between $f(a)$ and $f(b)$. Then there exists a number c in (a, b) such that

$$f(c) = k.$$

Bernard Bolzano

The first purely analytic proof of the Intermediate Value Theorem was given by the Bohemian mathematician and theologian Bernard Bolzano (1781–1848) in a paper published in 1817. Bolzano was one of the first mathematicians to begin to instil rigour into analysis. To quote Steve Russ, the leading expert on Bolzano's mathematics, this paper

> 'represents an important stage in the rigorous foundation of analysis and is one of the earliest occasions when the continuity of a function and the convergence of an infinite series are both defined and used correctly.'

Four years later, the theorem appeared in the *Cours d'Analyse* of Augustin-Louis Cauchy (1789–1857), but it is unlikely that Cauchy knew of Bolzano's work.

(Source: Russ, S.B. (1980) 'A Translation of Bolzano's Paper on the Intermediate Value Theorem', *Historia Mathematica*, vol. 7, no. 2, pp. 156–185.)

The Intermediate Value Theorem has two possible cases: we have either

$$f(a) < k < f(b) \quad \text{or} \quad f(a) > k > f(b).$$

These are illustrated in Figure 29.

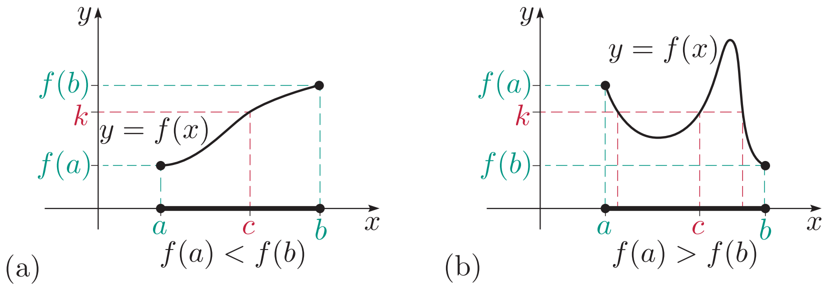


Figure 29 The two cases of the Intermediate Value Theorem

As the graph in Figure 29(b) shows, there may be more than one possible value of c such that $f(c) = k$.

The conclusion of the Intermediate Value Theorem may be *false* if f is discontinuous at even *one* point of $[a, b]$. For example, the function

$$f(x) = \begin{cases} 1/x, & 0 < |x| \leq 1, \\ 0, & x = 0, \end{cases} \tag{8}$$

is continuous on $[-1, 1]$ except at 0. For this function,

$$f(-1) = -1 \quad \text{and} \quad f(1) = 1,$$

but there is no number c in $(-1, 1)$ such that $f(c) = \frac{1}{2}$, as shown in Figure 30. The conclusion of the Intermediate Value Theorem may be also be false if $f(a) = f(b)$. For example, if $f(x) = x^2$ with $a = -1$ and $b = 1$, then $f(a) = f(b) = 1$ and the only possible value of k is 1, but there is no $c \in (-1, 1)$ such that $f(c) = 1$.

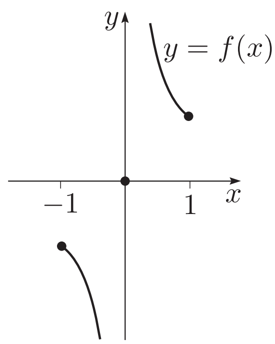


Figure 30 A function to which the Intermediate Value Theorem does not apply

Here is a typical application of the Intermediate Value Theorem.

Worked Exercise D50

Use the Intermediate Value Theorem to prove that there is a number c in $(0, 1)$ such that

$$c^5 + c - 1 = 0.$$

Solution

Consider the basic continuous function

$$f(x) = x^5 + x - 1.$$

Then f is continuous on $[0, 1]$ and also

$$f(0) = -1 \quad \text{and} \quad f(1) = 1.$$

Since $f(0) < 0 < f(1)$, it follows from the Intermediate Value Theorem that there is a number c in $(0, 1)$ such that

$$f(c) = 0; \quad \text{that is,} \quad c^5 + c - 1 = 0.$$

Notice that the function f is strictly increasing on $[0, 1]$, so in this case the number c must be *unique*.

To obtain further information about the location of the number c in Worked Exercise D50, we can use, for example, the **bisection method**. This involves repeatedly bisecting the interval containing the solution and determining the values of f at the bisection points, in order to find shorter and shorter intervals in which the number c must lie.

For example, at the first bisection the function

$$f(x) = x^5 + x - 1$$

satisfies

$$f\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^5 + \tfrac{1}{2} - 1 < 0 \quad \text{and} \quad f(1) = 1 > 0,$$

so the number c must lie in $\left(\frac{1}{2}, 1\right)$. To find an interval of length $\frac{1}{4}$ containing c, we next consider the value $f\left(\frac{3}{4}\right)$, and so on.

Exercise D67

Use the bisection method to find an interval of length $\frac{1}{16}$ containing the number c such that

$$c^5 + c - 1 = 0.$$

The bisection method can also be used to prove the Intermediate Value Theorem. We prove the special case of this result when $k = 0$ and $f(a) < f(b)$; the proof when $k = 0$ and $f(a) > f(b)$ is similar. The general case can then be deduced from the case when $k = 0$ by considering the function

$$F(x) = f(x) - k.$$

Theorem D53 Intermediate Value Theorem (special case)

Let f be a function continuous on $[a, b]$ and suppose that

$$f(a) < 0 < f(b).$$

Then there exists a number c in (a, b) such that

$$f(c) = 0.$$

Proof We use the bisection method.

First we define $[a_0, b_0] = [a, b]$ and let $p = \frac{1}{2}(a_0 + b_0)$, the midpoint of $[a_0, b_0]$. If $f(p) = 0$, then the proof is complete, since we can take $c = p$. Otherwise, we define

$$[a_1, b_1] = \begin{cases} [a_0, p], & \text{if } f(p) > 0, \\ [p, b_0], & \text{if } f(p) < 0. \end{cases}$$

In either case, we have

1. $[a_1, b_1] \subseteq [a_0, b_0]$
2. $b_1 - a_1 = \frac{1}{2}(b_0 - a_0)$
3. $f(a_1) < 0 < f(b_1)$.

Now we repeat this process, bisecting $[a_1, b_1]$ to obtain $[a_2, b_2]$, and so on. If, at any stage, we encounter a bisection point p such that $f(p) = 0$, then the proof is complete. Otherwise, we obtain a sequence of closed intervals

$$[a_n, b_n], \quad n = 0, 1, 2, \ldots,$$

with the properties that, for $n = 0, 1, 2, \ldots$,

1. $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$
2. $b_n - a_n = \left(\frac{1}{2}\right)^n (b_0 - a_0)$
3. $f(a_n) < 0 < f(b_n)$.

Property 1 implies that (a_n) is increasing and bounded above by b_0.

We now use the Monotone Convergence Theorem (Theorem D22 in Unit D2), which says that any sequence that is increasing and bounded above must be convergent.

Hence, by the Monotone Convergence Theorem, (a_n) is convergent. Let

$$\lim_{n\to\infty} a_n = c.$$

This is illustrated in Figure 31.

By property 2 and the Combination Rules for sequences,

$$\begin{aligned} \lim_{n\to\infty} b_n &= \lim_{n\to\infty} (a_n + (b_n - a_n)) \\ &= \lim_{n\to\infty} a_n + \lim_{n\to\infty} \left(\tfrac{1}{2}\right)^n (b_0 - a_0) \\ &= c + 0 = c. \end{aligned}$$

Now we use the fact that f is continuous at c to obtain

$$\lim_{n\to\infty} f(a_n) = f(c) \quad \text{and} \quad \lim_{n\to\infty} f(b_n) = f(c).$$

By property 3, $f(a_n) < 0$, for $n = 0, 1, 2, \ldots$, so $f(c) \leq 0$, by the Limit Inequality Rule (Theorem D11 in Unit D2). Likewise, $f(c) \geq 0$ because $f(b_n) > 0$, for $n = 0, 1, 2, \ldots$. Hence $f(c) = 0$, as required. ■

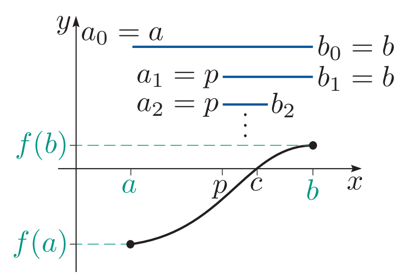


Figure 31 The limit c

3.2 Locating zeros of continuous functions

If f is a function and c is a real number such that

$$f(c) = 0,$$

then c is called a **zero** of the function f. We sometimes say that the function **vanishes** at c.

We often show that an equation has a solution by proving that a related continuous function has a zero (using the Intermediate Value Theorem with $k = 0$). You saw an example of this in Worked Exercise D50 and we give another one now.

Worked Exercise D51

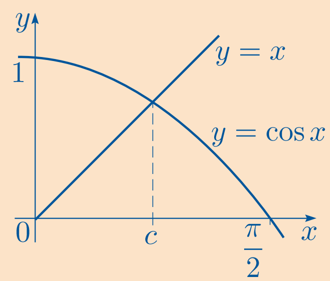
Prove that the equation

$$\cos x = x$$

has a solution in the interval $(0, 1)$.

Solution

The graphs shown below suggest that there is a solution to this equation.



We now prove that this is the case.

We consider the function

$$f(x) = \cos x - x$$

and show that f has a zero c in $(0, 1)$. Now f is continuous, by the Combination Rules. Moreover,

$$f(0) = \cos 0 - 0 = 1 > 0$$

and

$$f(1) = \cos 1 - 1 < 0.$$

Thus, by the Intermediate Value Theorem, there is a number c in $(0, 1)$ such that

$$f(c) = 0, \quad \text{so} \quad \cos c = c.$$

The result that was proved in Worked Exercise D51 is a special case of the result that you are asked to prove in the next exercise.

Exercise D68

Suppose that the function $f : [0,1] \longrightarrow [0,1]$ is continuous. Prove that the equation

$$f(x) = x$$

has a solution c in the interval $[0,1]$.

Hint: Consider the function $g(x) = f(x) - x$ $(x \in [0,1])$.

Zeros of polynomials

We now consider the problem of locating the zeros of polynomial functions. Recall that zeros of polynomials were discussed in Unit A2 *Number systems*, where you met the fact that a polynomial of degree n has at most n zeros.

First try the following exercise.

Exercise D69

Let

$$p(x) = x^6 - 4x^4 + x + 1 \quad (x \in \mathbb{R}).$$

Prove that p has a zero in each of the intervals $(-1,0), (0,1)$ and $(1,2)$.

When we wish to locate the zeros (if any) of a given polynomial, we can begin by applying the following result, which gives an interval in which the zeros *must* lie.

Theorem D54

Let

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \quad (x \in \mathbb{R}),$$

where $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$. Then all the zeros of p (if there are any) lie in the open interval $(-M, M)$, where

$$M = 1 + \max\{|a_{n-1}|, \ldots, |a_1|, |a_0|\}.$$

Proof Suppose we write

$$r(x) = \frac{p(x)}{x^n} - 1 = \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \quad (x \in \mathbb{R} - \{0\}),$$

so that

$$p(x) = x^n(1 + r(x)), \quad \text{for } x \neq 0. \tag{9}$$

Using the Triangle Inequality, we obtain

$$\begin{aligned} |r(x)| &= \left| \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right| \\ &\leq \left| \frac{a_{n-1}}{x} \right| + \cdots + \left| \frac{a_1}{x^{n-1}} \right| + \left| \frac{a_0}{x^n} \right| \\ &\leq (M-1) \left(\frac{1}{|x|} + \cdots + \frac{1}{|x|^{n-1}} + \frac{1}{|x|^n} \right). \end{aligned}$$

Thus if $|x| > 1$, so that $1/|x| < 1$, we have

$$\begin{aligned} |r(x)| &< (M-1) \left(\frac{1}{|x|} + \frac{1}{|x|^2} + \cdots \right) \\ &= (M-1)\frac{(1/|x|)}{1 - 1/|x|} = \frac{M-1}{|x|-1}, \end{aligned}$$

by summing the convergent geometric series.

Hence for $|x| \geq M$ we have $|r(x)| < 1$, and therefore $1 + r(x) > 0$. It now follows from equation (9) that $p(x)$ has the same sign as x^n for $|x| \geq M$. Since $x^n \neq 0$ for $|x| \geq M$, any zero of p must lie in the interval $(-M, M)$, as required. ■

The next worked exercise shows how we can use Theorem D54 in combination with the Intermediate Value Theorem to obtain information about the zeros of a polynomial.

Worked Exercise D52

Prove that the following polynomial has at least two zeros.

$$p(x) = 2x^4 - 4x^2 - 2x + 2 \quad (x \in \mathbb{R})$$

Solution

We first note that Theorem D54 can only be applied to polynomials for which the coefficient of x^n is equal to 1. So we find a polynomial that satisfies this condition and has the same zeros as p.

We have

$$p(x) = 2x^4 - 4x^2 - 2x + 2 = 2(x^4 - 2x^2 - x + 1).$$

For $q(x) = x^4 - 2x^2 - x + 1$ we have

$$M = 1 + \max\{|-2|, |-1|, |1|\} = 3,$$

and so all the zeros of q, and hence of p, lie in $(-3, 3)$, by Theorem D54.

We now compile a table of values of $p(n)$, for integers n in $[-3, 3]$.

n	-3	-2	-1	0	1	2	3
$p(n)$	134	22	2	2	-2	14	122

We find that $p(0)$ and $p(1)$ have opposite signs, as do $p(1)$ and $p(2)$. Thus, since p is continuous, it has a zero in each of the open intervals $(0, 1)$ and $(1, 2)$, by the Intermediate Value Theorem.

Thus we have proved that p has *at least* two zeros.

Since p is a polynomial of degree 4, we also know that it has *at most* 4 zeros. It can be shown that p has *exactly* two zeros, though we do not prove this here.

When using Theorem D54, it is often not necessary to calculate the values of $p(n)$ for *all* the integers n in the interval $[-M, M]$. For example, in Worked Exercise D52, in order to locate two zeros it would have been sufficient to calculate the values of $p(n)$ for n in $[-2, 2]$. Often it is a good idea to calculate the values of $p(n)$ for small values of n first and only calculate $p(n)$ for larger values if you have to; in other words, start filling in a table like that in the solution to Worked Exercise D52 in the middle and work as far outwards as necessary.

Exercise D70

Prove that the following polynomial has at least three zeros.

$$p(x) = x^5 + 3x^4 - x - 1 \quad (x \in \mathbb{R})$$

3.3 Extreme Value Theorem

We now describe another important property of continuous functions. First we give the following definitions.

Definitions

Let f be a function with domain A. Then

- f has **maximum value** $f(c)$ in A if $c \in A$ and
$$f(x) \leq f(c), \quad \text{for } x \in A$$
- f has **minimum value** $f(c)$ in A if $c \in A$ and
$$f(c) \leq f(x), \quad \text{for } x \in A$$
- f is **bounded** on A if, for some $M \in \mathbb{R}$,
$$|f(x)| \leq M, \quad \text{for } x \in A.$$

An **extreme value** is either a maximum or a minimum value.

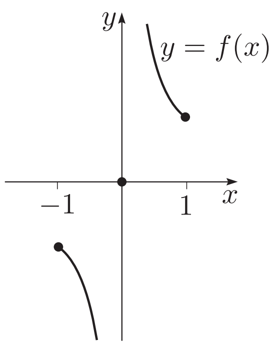


Figure 32 A function that is not bounded on its domain

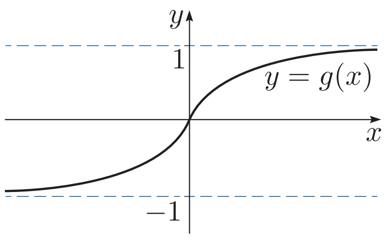


Figure 33 A bounded function with no extreme values

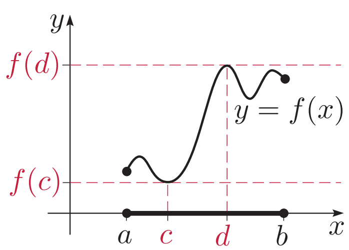


Figure 34 Extreme values of f

For example, the function $f(x) = \sin x$, with domain $\mathbb{R}$, has maximum value 1 in $\mathbb{R}$, since

$$\sin x \leq 1 = \sin\frac{\pi}{2}, \quad \text{for } x \in \mathbb{R}.$$

Also, this function f is bounded on $\mathbb{R}$, since

$$|\sin x| \leq 1, \quad \text{for } x \in \mathbb{R}.$$

On the other hand, the function

$$f(x) = \begin{cases} 1/x, & 0 < |x| \leq 1, \\ 0, & x = 0, \end{cases}$$

is not bounded on its domain $[-1, 1]$; see Figure 32.

Note that although a function that has both a maximum value and a minimum value is bounded, a function can be bounded without having a maximum value or a minimum value; for example, the function

$$g(x) = \frac{x}{1 + |x|}$$

is bounded on its domain $\mathbb{R}$, since

$$|g(x)| = \frac{|x|}{1 + |x|} < 1, \quad \text{for } x \in \mathbb{R},$$

but g is strictly increasing on $\mathbb{R}$, so it has no maximum or minimum value on $\mathbb{R}$. The graph of g is shown in Figure 33.

The following result states that a continuous function on a bounded closed interval always has a maximum value and a minimum value. It is illustrated in Figure 34.

Theorem D55 Extreme Value Theorem

Let f be a function continuous on $[a,b]$. Then there exist numbers c and d in $[a,b]$ such that

$$f(c) \le f(x) \le f(d), \quad \text{for } x \in [a,b].$$

An immediate consequence of the Extreme Value Theorem is the following corollary.

Corollary D56 Boundedness Theorem

Let f be a function continuous on $[a,b]$. Then there exists a number M such that

$$|f(x)| \le M, \quad \text{for } x \in [a,b].$$

The function

$$f(x) = \begin{cases} 1/x, & 0 < |x| \le 1, \\ 0, & x = 0, \end{cases}$$

whose graph was given in Figure 32, shows that the conclusions of the Extreme Value Theorem and the Boundedness Theorem may be *false* if f is discontinuous at even *one* point of $[a,b]$.

We now prove the Extreme Value Theorem. The proof illustrates the use of many of the techniques that you have met so far but if you are short of time you may prefer to skim through it, noting the main steps.

Proof of the Extreme Value Theorem We prove that there exists a number d in $[a,b]$ such that

$$f(x) \le f(d), \quad \text{for } x \in [a,b].$$

We use the function

$$g(x) = \frac{x}{1+|x|},$$

which is strictly increasing and continuous on $\mathbb{R}$, with

$$|g(x)| < 1, \quad \text{for } x \in \mathbb{R}.$$

The graph $y = g(x)$ was shown in Figure 33.

Then the function

$$h(x) = g(f(x)) \quad (x \in [a,b])$$

is continuous, by the Composition Rule, and

$$|h(x)| < 1, \quad \text{for } x \in [a,b].$$

The function h is easier to work with than the function f and, since g is strictly increasing, if we can find d such that $h(x) \leq h(d)$ for $x \in (a, b)$, then we can deduce that $f(x) \leq f(d)$ for $x \in [a, b]$.

Hence the image set $h([a, b])$ is bounded. Thus, by the Least Upper Bound Property of $\mathbb{R}$ which you met in Section 4 of Unit D1,

the supremum, M say, of $h([a, b])$ exists.

Remember that the least upper bound of a set A of real numbers is also called its supremum, and is denoted by $\sup A$.

We now use the bisection method to find $d \in [a, b]$ such that $h(d) = M$.

We define $[a_0, b_0] = [a, b]$ and $p = \frac{1}{2}(a_0 + b_0)$. Then at least one of the image sets $h([a_0, p])$ and $h([p, b_0])$ must have least upper bound M.

We now choose the interval $[a_1, b_1]$ to be whichever of $h([a_0, p])$ and $h([p, b_0])$ has least upper bound M, or either if both do.

Thus we can choose $[a_1, b_1]$ such that

1. $[a_1, b_1] \subseteq [a_0, b_0]$
2. $b_1 - a_1 = \frac{1}{2}(b_0 - a_0)$
3. $M = \sup h([a_1, b_1])$.

Now we repeat this process, bisecting $[a_1, b_1]$ to obtain $[a_2, b_2]$, and so on. We obtain a sequence of closed intervals

$$[a_n, b_n], \quad n = 0, 1, 2, \ldots,$$

with the following properties for $n = 0, 1, 2, \ldots$:

1. $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$
2. $b_n - a_n = \left(\frac{1}{2}\right)^n (b_0 - a_0)$
3. $M = \sup h([a_n, b_n])$.

As in the proof of the Intermediate Value Theorem, properties 1 and 2 imply that there is a real number $d \in [a, b]$ such that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = d.$$

By property 3, for each $n = 1, 2, \ldots$, there is a number t_n such that

$$a_n \leq t_n \leq b_n \quad \text{and} \quad M - 1/n \leq h(t_n) \leq M,$$

because each number $M - 1/n$ is *not* an upper bound of the image set $h([a, b])$.

Hence, by the Squeeze Rule for sequences,

$$\lim_{n\to\infty} t_n = d \quad \text{and} \quad \lim_{n\to\infty} h(t_n) = M.$$

Thus, by the continuity of h at d,

$$h(d) = \lim_{n\to\infty} h(t_n) = M,$$

so

$$h(x) = g(f(x)) \leq g(f(d)) = h(d), \quad \text{for } x \in [a, b].$$

Since g is strictly increasing, it follows that

$$f(x) \leq f(d), \quad \text{for } x \in [a, b],$$

as required.

Similar reasoning shows that there exists $c \in [a, b]$ such that

$$f(c) \leq f(x), \quad \text{for } x \in [a, b].$$

Antipodal Points Theorem (optional)

We conclude this section by presenting a corollary of the Intermediate Value Theorem. This application is included for your interest and is not assessed.

We ask whether there must always be a pair of *antipodal points* (that is, points which lie at opposite ends of a line segment through the centre of the Earth) on the equator at which the temperature is the same. (The equator is shown in Figure 35.) The following result uses the Intermediate Value Theorem to show that the answer is 'yes'. Here $g(\theta)$ represents the temperature at a point on the equator at an angle θ radians east of the Greenwich meridian. If $g(c) = g(c + \pi)$, then c and $c + \pi$ represent antipodal points with the same temperature.

Figure 35 The equator

Theorem D57 Antipodal Points Theorem

If $g : [0, 2\pi] \longrightarrow \mathbb{R}$ is a continuous function and $g(0) = g(2\pi)$, then there exists a number c in $[0, \pi]$ such that

$$g(c) = g(c + \pi).$$

Proof The theorem is illustrated in Figure 36 below.

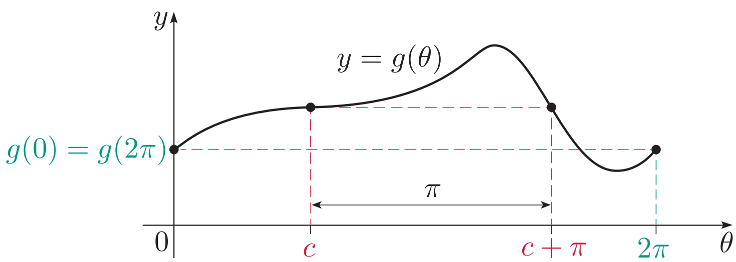


Figure 36 The Antipodal Points Theorem

First note that if $g(0) = g(\pi)$, then we can take $c = 0$. So let us assume that

$$g(0) < g(\pi). \tag{10}$$

(The proof in the case $g(0) > g(\pi)$ is similar.)

Now we define

$$h(\theta) = g(\theta + \pi) \quad (\theta \in [0, \pi]),$$

and consider the graphs $y = g(\theta)$ and $y = h(\theta)$, for $0 \le \theta \le \pi$, shown in Figure 37. The graph $y = h(\theta)$ is obtained by translating to the left the part of the graph $y = g(\theta)$ corresponding to $\pi \le \theta \le 2\pi$.

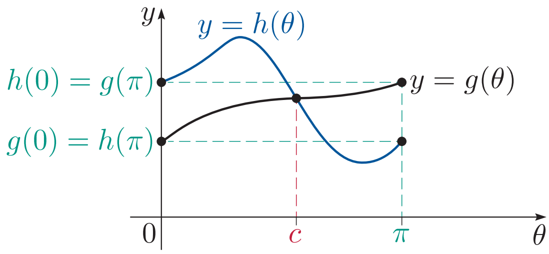


Figure 37 The graphs of g and h

Karol Borsuk

Stanislaw Ulam

Since

$$h(0) = g(0 + \pi) = g(\pi) \quad \text{and} \quad h(\pi) = g(\pi + \pi) = g(0),$$

inequality (10) can be rewritten as

$$g(0) < h(0) \quad \text{and} \quad g(\pi) > h(\pi).$$

This suggests that our two graphs must cross at some point c in $(0, \pi)$, giving

$$g(c) = h(c) \quad \text{and hence} \quad g(c) = g(c + \pi).$$

To make this argument rigorous, we define a function f as

$$f(\theta) = g(\theta) - h(\theta) \quad (\theta \in [0, \pi]),$$

which is continuous on $[0, \pi]$. Also,

$$f(0) = g(0) - h(0) < 0 \quad \text{and} \quad f(\pi) = g(\pi) - h(\pi) > 0.$$

Thus, by the Intermediate Value Theorem with $k = 0$, there exists a number c in $(0, \pi)$ such that $f(c) = 0$, so $g(c) = h(c)$ and hence

$$g(c) = g(c + \pi),$$

as required. ■

The Antipodal Points Theorem is the one-dimensional case of the Borsuk–Ulam theorem, an important result about continuous functions that holds in any number of dimensions. It is named after the Polish mathematicians Karol Borsuk (1905–1982) and Stanislaw Ulam (1909–1984) – Ulam was the first to formulate the theorem and in 1933 Borsuk was the first to prove it. The two-dimensional case of the theorem can be illustrated by saying that at any moment there is always pair of antipodal points on the Earth's surface with equal temperatures and barometric pressures.

4 Inverse functions

In this section you will meet the Inverse Function Rule, which gives conditions for a continuous function f to have a continuous inverse function f^{-1}. You will then see how the Inverse Function Rule can be used to show that the inverse functions of various standard functions exist and are continuous. Finally, you will see how to use the exponential function and its inverse function to define a^x, for $a > 0$ and *all* $x \in \mathbb{R}$.

4.1 Inverse Function Rule

At the end of Section 1 we discussed the function

$$f(x) = x^5 + x - 1 \quad (x \in \mathbb{R}).$$

We showed that f is strictly increasing and hence one-to-one, but we could not prove that $f(\mathbb{R}) = \mathbb{R}$, so we could not prove that the inverse function f^{-1} has domain $\mathbb{R}$. In this section you will see that f^{-1} does indeed have domain $\mathbb{R}$ and, moreover, f^{-1} is continuous on $\mathbb{R}$. We now know that f is continuous (as it is a polynomial) and this means that we can apply a result known as the *Inverse Function Rule*. The proof of this result is based on the Intermediate Value Theorem and is given in Subsection 4.4. The graphs of the functions in the statement of the theorem are shown in Figures 38 and 39.

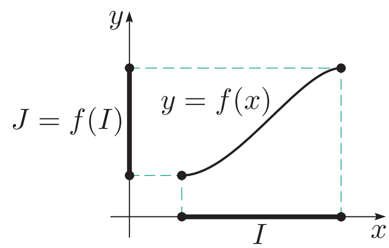


Figure 38 The graph of $f : I \longrightarrow J$

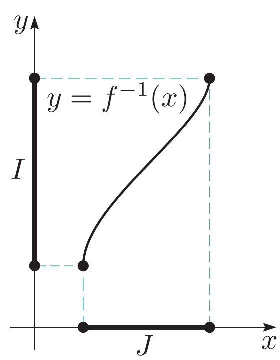


Figure 39 The graph of $f^{-1} : J \longrightarrow I$

Theorem D58 Inverse Function Rule

Let $f : I \longrightarrow J$, where I is an interval and J is the image set $f(I)$, be a function such that

1. f is strictly increasing on I
2. f is continuous on I.

Then J is an interval and f has an inverse function $f^{-1} : J \longrightarrow I$ such that

1′. f^{-1} is strictly increasing on J

2′. f^{-1} is continuous on J.

Remarks

1. The interval I can be *any* type of interval: open or closed, half-open, bounded or unbounded.
2. There is another version of the Inverse Function Rule with 'strictly increasing' replaced by 'strictly decreasing'.

If we return to the function

$$f(x) = x^5 + x - 1 \quad (x \in \mathbb{R}),$$

then this satisfies the conditions of the Inverse Function Rule with $I = \mathbb{R}$. So we can deduce that f has a *continuous* inverse function and that the domain of this inverse function is an interval $J = f(I) = f(\mathbb{R})$, but we still need to determine what this interval is.

In general, in order to identify the interval J which arises in the Inverse Function Rule, it is sufficient to determine the **endpoints** of J, which may be real numbers or one of the symbols ∞ and $-\infty$. For example, $(0, 1]$ has endpoints 0 and 1, and $[1, \infty)$ has endpoints 1 and ∞. (As earlier, do not let this use of the *symbol* ∞ tempt you to think that ∞ is a real number.) We must also determine whether or not these endpoints *belong* to J. Figure 40 illustrates various cases that can occur.

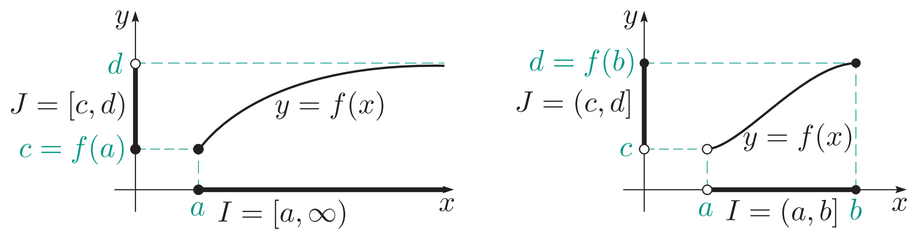


Figure 40 Two examples of the intervals I and J in the Inverse Function Rule

If a is an endpoint of I and $a \in I$, then it follows from the fact that f is increasing that $c = f(a)$ is the corresponding endpoint of J and $c \in J$.

On the other hand, if a is an endpoint of I and $a \notin I$ (this includes the possibility that a may be ∞ or $-\infty$), then it is a little harder to find the corresponding endpoint c of J. We will show in Subsection 4.4 that, in this case, if (a_n) is a monotonic sequence in I and $a_n \to a$, then $f(a_n) \to c$. This leads to the following strategy for establishing that a function has a continuous inverse function with a specified domain.

Strategy D15

To prove that $f : I \longrightarrow J$, where I is an interval with endpoints a and b, has a continuous inverse $f^{-1} : J \longrightarrow I$, do the following.

1. Show that f is strictly increasing on I.
2. Show that f is continuous on I.
3. Determine the endpoint c of J corresponding to the endpoint a of I as follows:
   - if $a \in I$, then $c = f(a)$ and $c \in J$
   - if $a \notin I$, then $f(a_n) \to c$ and $c \notin J$, where (a_n) is a monotonic sequence in I such that $a_n \to a$.

   Determine the endpoint d of J, corresponding to the endpoint b of I, similarly.

Note that there is a corresponding version of Strategy D15 if f is strictly decreasing. In the strictly increasing version the left endpoint of I corresponds to the left endpoint of J, whereas in the strictly decreasing version the left endpoint of I corresponds to the right endpoint of J.

Before returning to our original example, we first apply this strategy in a more straightforward case, where the domain of the function is a closed bounded interval.

Worked Exercise D53

Prove that the function

$$f(x) = x^4 + 2x + 3 \quad (x \in [0, 2])$$

has a continuous inverse function, with domain $[3, 23]$.

Solution

We use Strategy D15.

1. We showed that f is strictly increasing on $[0, \infty)$ in Exercise D57(a).

2. The function

$$f(x) = x^4 + 2x + 3 \quad (x \in [0, 2])$$

is the restriction to $[0, 2]$ of a polynomial which is continuous on $\mathbb{R}$. Hence f is continuous.

We have shown that f satisfies the conditions of the Inverse Function Rule and so f has a continuous inverse function with domain $J = f([0, 2])$. We now use step 3 of the strategy to determine J by finding its endpoints. In this case, both the endpoints of the domain of f are in the domain (since it is a closed interval) and so we can find the endpoints of J by finding the images of the endpoints of $[0, 2]$.

3. Since $f(0) = 3$ and $f(2) = 23$, we have $f([0, 2]) = [3, 23]$.

So, by the Inverse Function Rule, f has a continuous inverse function

$$f^{-1} : [3, 23] \longrightarrow [0, 2].$$

The domain of the function is important here and we could not define an inverse function if the domain was the whole of $\mathbb{R}$.

We now return to our original example.

Worked Exercise D54

Prove that the function

$$f(x) = x^5 + x - 1 \quad (x \in \mathbb{R})$$

has a continuous inverse function, with domain $\mathbb{R}$.

Solution

We use Strategy D15.

1. We showed that f is strictly increasing in Worked Exercise D41.
2. The function f is continuous as it is a polynomial.

We have shown that f has a continuous inverse function with domain $f(\mathbb{R})$. We now apply step 3 of the strategy to find $f(\mathbb{R})$. In this case the interval is unbounded and so we have to find one monotonic sequence tending to ∞ and another monotonic sequence tending to $-\infty$ and then investigate the behaviour of the images of these sequences as $n \to \infty$.

3. We first choose the increasing sequence (n) which tends to ∞, the right endpoint of $\mathbb{R}$. Then

$$f(n) = n^5 + n - 1 \to \infty \text{ as } n \to \infty.$$

Thus the right endpoint of $f(\mathbb{R})$ is ∞. We now choose the decreasing sequence $(-n)$ which tends to $-\infty$, the left endpoint of $\mathbb{R}$. Then

$$f(-n) = -n^5 - n - 1 \to -\infty \text{ as } n \to \infty.$$

So the left endpoint of $f(\mathbb{R})$ is $-\infty$. Thus $f(\mathbb{R}) = \mathbb{R}$.

It follows from the Inverse Function Rule that f has a continuous inverse function

$$f^{-1} : \mathbb{R} \longrightarrow \mathbb{R}.$$

The following exercise gives you the opportunity to practise using Strategy D15.

Exercise D71

Prove that the function

$$f(x) = x^2 - \frac{1}{x} \quad (x \in (0, \infty))$$

has a continuous inverse function with domain $\mathbb{R}$.

Hint: Use the solution to Exercise D57(b).

4.2 Inverses of standard functions

We now use the Inverse Function Rule to define continuous inverse functions for various standard functions. You are already familiar with these inverse functions, but we can now *prove* that they exist and are continuous. For each function, we give brief remarks on the three steps of Strategy D15. We also revise some of the properties of these inverse functions. Here we often use the fact that the restriction of a continuous function is continuous, mentioned at the end of Subsection 2.2.

nth root function

We asserted the existence of the nth root function in Section 5 of Unit D1. We can now provide a proof of that result. The graph of $f(x) = x^n$ for $x \in [0, \infty)$ is shown in Figure 41.

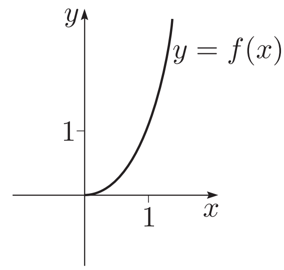


Figure 41 $f(x) = x^n \quad (x \in [0, \infty))$

nth root function

For any positive integer $n \geq 2$, the function

$$f(x) = x^n \quad (x \in [0, \infty))$$

has a strictly increasing continuous inverse function $f^{-1}(x) = \sqrt[n]{x}$, with domain $[0, \infty)$ and image set $[0, \infty)$, called the ***n*th root function**.

We follow the steps of Strategy D15.

1. f is strictly increasing on $[0, \infty)$.
2. f is continuous on $[0, \infty)$.
3. $f(0) = 0$, and $f(k) = k^n \to \infty$ as $k \to \infty$, so

$$f([0, \infty)) = [0, \infty).$$

(We use k here, to avoid using n for two different purposes in the same expression.)

Hence f has a strictly increasing continuous inverse function

$$f^{-1} : [0, \infty) \longrightarrow [0, \infty).$$

The graph of f^{-1} is shown in Figure 42.

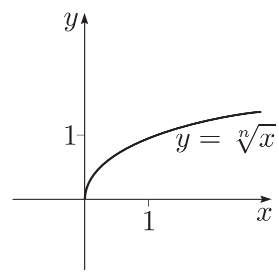


Figure 42 The nth root function

Inverse trigonometric functions

$\sin^{-1}$

The function

$$f(x) = \sin x \quad \left(x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$$

has a strictly increasing continuous inverse function, with domain $[-1, 1]$ and image set $[-\pi/2, \pi/2]$, called $\mathbf{sin^{-1}}$.

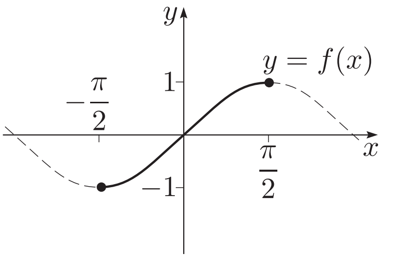


Figure 43 $f(x) = \sin x \quad (x \in [-\pi/2, \pi/2])$

The graph of $f(x) = \sin x$ for $x \in [-\pi/2, \pi/2]$ is shown in Figure 43.

We follow the steps of Strategy D15.

1. The geometric definition of $f(x) = \sin x$ shows that f is strictly increasing on $[-\pi/2, \pi/2]$.
2. f is continuous on $[-\pi/2, \pi/2]$.
3. $\sin(-\pi/2) = -1$ and $\sin(\pi/2) = 1$, so
$$f\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right) = [-1, 1].$$

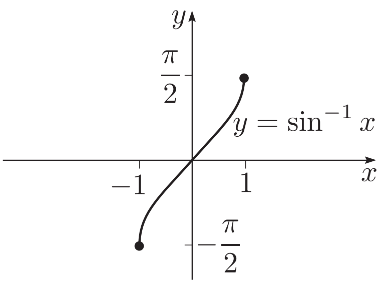


Figure 44 The inverse sine function

Hence f has a strictly increasing continuous inverse function

$$f^{-1} : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

The graph of f^{-1} is shown in Figure 44.

The decreasing version of Strategy D15 can be applied similarly to prove that the cosine function has an inverse, if we restrict its domain suitably. The graph of $f(x) = \cos x$ for $x \in [0, \pi]$ is shown in Figure 45.

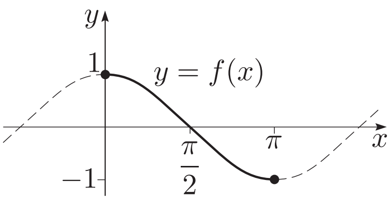


Figure 45 $f(x) = \cos x \quad (x \in [0, \pi])$

$\cos^{-1}$

The function

$$f(x) = \cos x \quad (x \in [0, \pi])$$

has a strictly decreasing continuous inverse function, with domain $[-1, 1]$ and image set $[0, \pi]$, called $\mathbf{cos^{-1}}$.

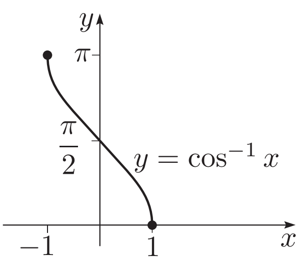


Figure 46 The inverse cosine function

The graph of f^{-1} is shown in Figure 46. The domain $[0, \pi]$ of f is chosen, by convention, so that f is a restriction of the cosine function which is strictly decreasing and continuous.

Similarly, to form an inverse of the tangent function, we restrict its domain to $(-\pi/2, \pi/2)$, since the tangent function is strictly increasing and continuous on this interval. The graph of the tangent function on this interval is shown in Figure 47.

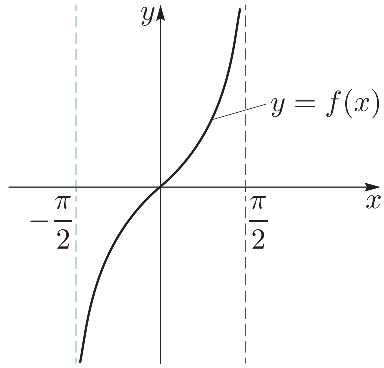


Figure 47 $f(x) = \tan x \ (x \in (-\pi/2, \pi/2))$

$\tan^{-1}$

The function

$$f(x) = \tan x \quad \left(x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$$

has a strictly increasing continuous inverse function, with domain $\mathbb{R}$ and image set $(-\pi/2, \pi/2)$, called $\mathbf{tan^{-1}}$.

In this case, the image set $f((-\pi/2, \pi/2))$ is $\mathbb{R}$ because if (a_n) is a monotonic sequence in $(-\pi/2, \pi/2)$ and $a_n \to \pi/2$ as $n \to \infty$, then

$$f(a_n) = \tan a_n = \frac{\sin a_n}{\cos a_n} \to \infty \quad \text{as } n \to \infty.$$

The graph of f^{-1} is shown in Figure 48.

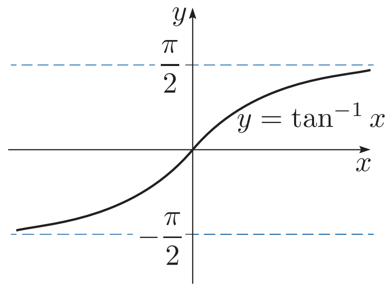


Figure 48 The inverse tangent function

Note that some texts use arcsin, arccos and arctan instead of $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$, respectively. These names arise from the geometric definitions of the trigonometric functions.

Exercise D72

(a) Determine the values of

$$\sin^{-1}(1/\sqrt{2}), \quad \cos^{-1}\left(-\tfrac{1}{2}\right) \quad \text{and} \quad \tan^{-1}(\sqrt{3}).$$

(b) Prove that

$$\cos(2\sin^{-1} x) = 1 - 2x^2, \quad \text{for } x \in [1, 1].$$

Hint: Let $y = \sin^{-1} x$ and use a suitable trigonometric identity from the module Handbook.

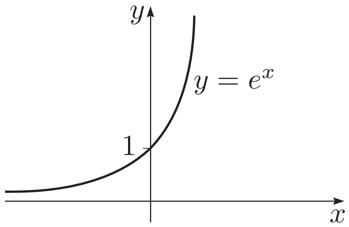


Figure 49 The exponential function

Inverse function of the exponential function

We now discuss one of the most important inverse functions: the inverse of the exponential function. The graph of the exponential function is shown in Figure 49.

log

The function

$$f(x) = e^x \quad (x \in \mathbb{R})$$

has a strictly increasing continuous inverse function f^{-1}, with domain $(0, \infty)$ and image set $\mathbb{R}$, called **log** or **ln**.

We follow the steps of Strategy D15.

1. f is strictly increasing on $\mathbb{R}$, since

$$\begin{aligned} x_1 < x_2 &\implies x_2 - x_1 > 0 \\ &\implies e^{x_2 - x_1} > 1 \quad (\text{since } e^x > 1 + x > 1, \text{ for } x > 0) \\ &\implies e^{x_2}/e^{x_1} > 1 \\ &\implies e^{x_2} > e^{x_1}. \end{aligned}$$

2. f is continuous on $\mathbb{R}$.
3. Since

$$f(n) = e^n \to \infty \text{ as } n \to \infty,$$

and

$$f(-n) = e^{-n} = (1/e)^n \to 0 \text{ as } n \to \infty,$$

the image set $f(\mathbb{R}) = (0, \infty)$.

Hence f has a strictly increasing continuous inverse function

$$f^{-1} : (0, \infty) \longrightarrow \mathbb{R}.$$

The graph of f^{-1} is shown in Figure 50.

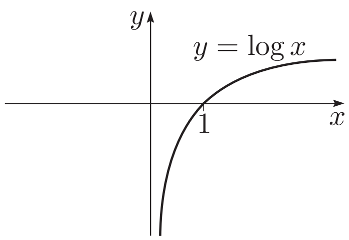


Figure 50 The log function

Exercise D73

Prove that

$$\log(xy) = \log x + \log y, \quad \text{for } x, y \in (0, \infty).$$

Hint: Let $a = \log x$ and $b = \log y$.

Inverse hyperbolic functions

We end this subsection by considering the inverse hyperbolic functions. We first consider the sinh function which is shown in Figure 51.

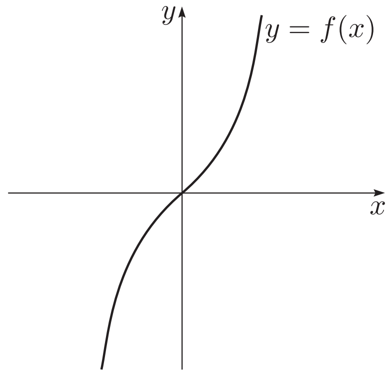


Figure 51
$f(x) = \sinh x \ (x \in \mathbb{R})$

$\sinh^{-1}$

The function

$$f(x) = \sinh x = \tfrac{1}{2}(e^x - e^{-x}) \quad (x \in \mathbb{R})$$

has a strictly increasing continuous inverse function f^{-1}, with domain $\mathbb{R}$ and image set $\mathbb{R}$, called $\mathbf{sinh^{-1}}$.

We follow the steps of Strategy D15.

1. f is strictly increasing on $\mathbb{R}$, since both the functions

   $$x \longmapsto e^x \quad \text{and} \quad x \longmapsto -e^{-x}$$

   are strictly increasing on $\mathbb{R}$.
2. f is continuous on $\mathbb{R}$, by the Combination Rules.
3. Since

   $$f(n) = \tfrac{1}{2}(e^n - e^{-n}) \to \infty \text{ as } n \to \infty,$$

   and

   $$f(-n) = \tfrac{1}{2}(e^{-n} - e^n) \to -\infty \text{ as } n \to \infty,$$

   the image set $f(\mathbb{R}) = \mathbb{R}$.

Hence f has a strictly increasing continuous inverse function

$$f^{-1} : \mathbb{R} \longrightarrow \mathbb{R}.$$

The graph of f^{-1} is shown in Figure 52.

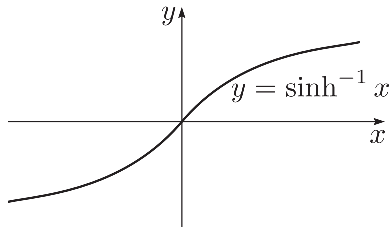


Figure 52 The inverse sinh function

We next consider the cosh function for $x \in [0, \infty)$. The graph is shown in Figure 53.

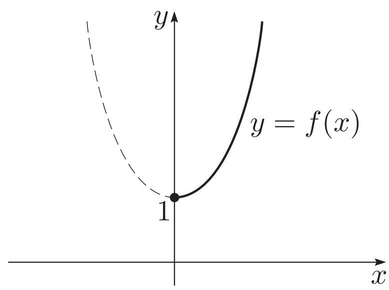


Figure 53
$f(x) = \cosh x \quad (x \in [0, \infty))$

$\cosh^{-1}$

The function

$$f(x) = \cosh x = \tfrac{1}{2}(e^x + e^{-x}) \quad (x \in [0, \infty))$$

has a strictly increasing continuous inverse function f^{-1}, with domain $[1, \infty)$ and image set $[0, \infty)$, called $\mathbf{cosh^{-1}}$.

We again follow the steps of Strategy D15.

1. f is strictly increasing on $[0,\infty)$, since $\cosh x = (1+\sinh^2 x)^{1/2}$ and the function $x \longmapsto \sinh x$ is strictly increasing on $[0,\infty)$.
2. f is continuous on $[0,\infty)$, by the Combination Rules.
3. Since $f(0) = 1$ and
$$f(n) = \tfrac{1}{2}(e^n + e^{-n}) \to \infty \text{ as } n \to \infty,$$
the image set $f([0,\infty)) = [1,\infty)$.

Hence f has a strictly increasing continuous inverse function
$$f^{-1} : [1,\infty) \longrightarrow [0,\infty).$$

The graph of f^{-1} is shown in Figure 54.

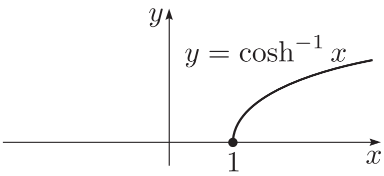


Figure 54 The inverse cosh function

Strategy D15 can be applied in a similar way to show that the function $f(x) = \tanh x$ is strictly increasing and continuous on $\mathbb{R}$, with $f(\mathbb{R}) = (-1,1)$. The graph of this function is shown in Figure 55.

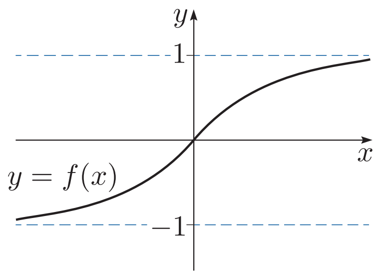


Figure 55 $f(x) = \tanh x \quad (x \in \mathbb{R})$

$\tanh^{-1}$

The function
$$f(x) = \tanh x = \frac{\sinh x}{\cosh x} \quad (x \in \mathbb{R})$$
has a strictly increasing continuous inverse function f^{-1}, with domain $(-1,1)$ and image set $\mathbb{R}$, called $\mathbf{tanh^{-1}}$.

The graph of the inverse tanh function is shown in Figure 56.

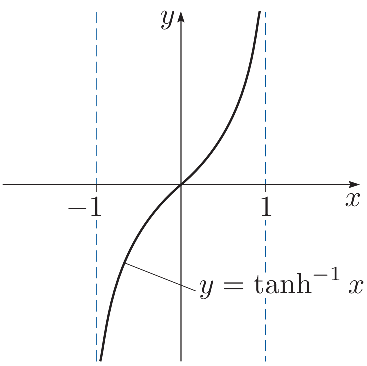


Figure 56 The inverse tanh function

The inverse hyperbolic functions can all be expressed in terms of log, as we show for $\sinh^{-1}$ in the following worked exercise.

Worked Exercise D55

Prove that
$$\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right), \quad \text{for } x \in \mathbb{R}.$$

Solution

Let $y = \sinh^{-1} x$, for $x \in \mathbb{R}$, so
$$x = \sinh y = \tfrac{1}{2}(e^y - e^{-y}).$$

Multiplying both sides by e^y and rearranging, we obtain
$$e^{2y} - 2xe^y - 1 = (e^y)^2 - 2xe^y - 1 = 0.$$

This is a quadratic equation in e^y, with solution
$$e^y = x \pm \sqrt{x^2+1}.$$

Since $e^y > 0$, we must choose the + sign. We obtain

$$y = \sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right).$$

Exercise D74

Prove that

$$\cosh^{-1} x = \log\left(x + \sqrt{x^2-1}\right), \quad \text{for } x \in [1,\infty).$$

4.3 Defining exponential functions

In this subsection we consider how to properly define *exponential functions*, by which we mean functions whose rule is of the form $f(x) = a^x$ for some $a > 0$. (As you have seen, when $a = e$ we refer to the function as *the* exponential function.)

In the introduction to Book D we asked a question about the graph of $y = 2^x$, shown in Figure 57.

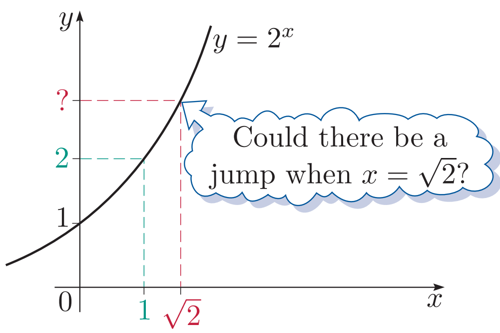


Figure 57 The graph of $y = 2^x$

In Unit D1 we defined the expression a^x for $a > 0$ when x is a rational number, but *not* when x is irrational. We now provide this missing definition, and also prove that the resulting function $x \longmapsto a^x$ is continuous. In particular, it follows that the graph of $y = 2^x$ cannot have any gaps.

Recall, from Section 4 of Unit D3, that we define

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{n!} \quad (x \geq 0)$$

and

$$e^x = \left(e^{-x}\right)^{-1} \quad (x < 0).$$

As we saw in Subsection 4.2, the function $x \longmapsto e^x$ is strictly increasing and continuous, and has a strictly increasing continuous inverse function

$$x \longmapsto \log x \quad (x \in (0, \infty)).$$

You saw in Exercise D73 that the function log has the property that

$$\log(ab) = \log a + \log b, \quad \text{for } a, b \in (0, \infty).$$

Thus, if $a > 0$ and $n \in \mathbb{N}$, then

$$\log(a^n) = n \log a, \quad \text{so} \quad a^n = e^{n \log a}.$$

With a little further manipulation, we can show that this equation for a^n remains true if n is replaced by any rational number x. Thus it makes sense to *define* a^x, for $a > 0$ and x irrational, by using this equation.

Definition

If $a > 0$, then

$$a^x = e^{x \log a} \quad (x \in \mathbb{R}).$$

For example,

$$2^\pi = e^{\pi \log 2}.$$

With this definition of a^x, we can verify that the function $x \longmapsto a^x$ is continuous. This follows immediately from the continuity of the function $x \longmapsto e^x$ and the Composition Rule. Moreover, we can also deduce the usual Index Laws for a^x from those for e^x. We state these below without proof.

Theorem D59

(a) If $a > 0$, then the function

$$x \longmapsto a^x = e^{x \log a} \quad (x \in \mathbb{R})$$

is continuous.

(b) If $a, b > 0$ and $x, y \in \mathbb{R}$, then

$$a^x b^x = (ab)^x, \quad a^x a^y = a^{x+y} \quad \text{and} \quad (a^x)^y = a^{xy}.$$

In particular, it follows from Theorem D59(b) that manipulations such as

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \times \sqrt{2}} = \sqrt{2}^2 = 2 \tag{11}$$

are justified.

Equation (11) gives an unexpected proof of the result that

there exist irrational numbers a and b such that a^b is rational.

For if $\sqrt{2}^{\sqrt{2}}$ is rational, then we can take $a = b = \sqrt{2}$, but if $\sqrt{2}^{\sqrt{2}}$ is irrational, then (by equation (11)) we can take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. In fact, it can be shown using more complicated arguments that $\sqrt{2}^{\sqrt{2}}$ is irrational, but we do not prove this here.

The number $\sqrt{2}^{\sqrt{2}} = \sqrt{2^{\sqrt{2}}}$ is the square root of the number known as the Gelfond–Schneider constant $2^{\sqrt{2}}$.

In his address at the International Congress of Mathematicians in Paris in 1900, David Hilbert (1862–1943) gave his famous list of mathematical problems, the seventh of which asked for a proof that α^β is an irrational transcendental number when α is algebraic and not equal to 0 or 1, and β is irrational and algebraic. (A number is *algebraic* if it is the root of a non-zero polynomial equation with integer coefficients and *transcendental* if it is not.) He included as a particular example the number $2^{\sqrt{2}}$. In fact, Hilbert believed that proving the transcendence of $2^{\sqrt{2}}$ was such a hard problem that he said in a lecture in 1919 that he thought nobody present would live to see it proved. He turned out to be very wrong! In 1929, the Russian mathematician Aleksandr Gelfond (1906–1968) proved Hilbert's Seventh Problem for the special case where β is a quadratic irrational, which includes $2^{\sqrt{2}}$. And in 1934 Gelfond went further and obtained the general solution, as did the German mathematician Theodor Schneider (1911–1988) independently later the same year.

Aleksandr Gelfond

Theodor Schneider

Exercise D75

Use the definition of a^x to prove that each of the following functions is continuous.

(a) $f(x) = x^\alpha \quad (x \in (0, \infty))$, where α is any fixed real number

(b) $f(x) = x^x \quad (x \in (0, \infty))$

4.4 Proof of the Inverse Function Rule

In this subsection we prove the Inverse Function Rule and we also justify step 3 of Strategy D15 for finding the domain of the inverse function. The proofs use many of the techniques that you have met so far, but if you are short of time you may prefer to skim read them. The graphs of the function and its inverse in the statement of the Inverse Function Rule are shown in Figures 58 and 59.

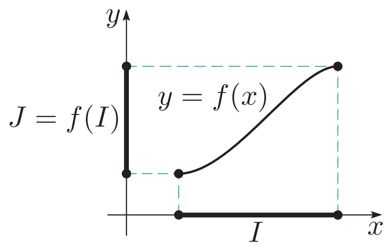


Figure 58 The graph of $f : I \longrightarrow J$

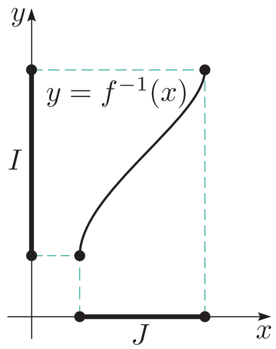


Figure 59 The graph of $f^{-1} : J \longrightarrow I$

Theorem D58 Inverse Function Rule

Let $f : I \longrightarrow J$, where I is an interval and J is the image set $f(I)$, be a function such that

1. f is strictly increasing on I
2. f is continuous on I.

Then J is an interval and f has an inverse function $f^{-1} : J \longrightarrow I$, such that

1′. f^{-1} is strictly increasing on J

2′. f^{-1} is continuous on J.

Proof First we prove that $J = f(I)$ is an interval. Suppose that $y_1, y_2 \in f(I)$ with $y_1 < y_2$ and that y is any number in the interval (y_1, y_2). To show that J is an interval, we must prove that $y \in f(I)$.

Now $y_1 = f(x_1)$ and $y_2 = f(x_2)$ for some $x_1, x_2 \in I$, with $x_1 < x_2$ because f is strictly increasing. Thus, since f is continuous, it follows from the Intermediate Value Theorem that there is a number $x \in (x_1, x_2)$ such that $f(x) = y$. Hence $y \in f(I)$, as required.

Next, the function f is strictly increasing and is therefore one-to-one. Thus $f^{-1} : J \longrightarrow I$ exists, where $J = f(I)$.

To prove that f^{-1} is strictly increasing on J, we have to show that

$$y_1 < y_2 \implies f^{-1}(y_1) < f^{-1}(y_2), \quad \text{for } y_1, y_2 \in J.$$

We use proof by contraposition.

This implication holds because, for $y_1, y_2 \in J$, we have

$$\begin{aligned} f^{-1}(y_1) \geq f^{-1}(y_2) &\implies f(f^{-1}(y_1)) \geq f(f^{-1}(y_2)) \\ &\implies y_1 \geq y_2. \end{aligned}$$

Finally, we prove that f^{-1} is continuous on J. Let $y \in J$ and assume, for simplicity, that y is not an endpoint of J.

Only a slight modification to the argument is needed if y is an endpoint of J.

Then $y = f(x)$ for some $x \in I$, and we want to prove that

$$y_n \to y \implies f^{-1}(y_n) \to f^{-1}(y) = x.$$

Thus we assume that $y_n \to y$ and we want to deduce that

for each $\epsilon > 0$, there is an integer N such that

$$x - \epsilon < f^{-1}(y_n) < x + \epsilon, \quad \text{for all } n > N. \tag{12}$$

By taking ϵ small enough, we can assume that $x - \epsilon \in I$ and $x + \epsilon \in I$.

Now since f is strictly increasing, we know that

$$f(x - \epsilon) < f(x) < f(x + \epsilon).$$

Also, because $y_n \to y = f(x)$, there is an integer N such that

$$f(x - \epsilon) < y_n < f(x + \epsilon), \quad \text{for all } n > N.$$

Thus, since f^{-1} is a strictly increasing function, we obtain inequalities (12).

This completes the proof of the Inverse Function Rule. ■

Finally, we justify step 3 of Strategy D15 for finding the endpoints of $J = f(I)$.

Strategy D15, step 3

Let f satisfy the assumptions of the Inverse Function Rule and let a be an endpoint of I. Determine the corresponding endpoint c of J as follows:

- if $a \in I$, then $c = f(a)$ and $c \in J$
- if $a \notin I$, then $f(a_n) \to c$ and $c \notin J$, where (a_n) is a monotonic sequence in I such that $a_n \to a$.

Proof Suppose that a is the left endpoint of I; the argument for the right endpoint is similar.

If $a \in I$, then $c = f(a) \in J$ and (since f is an increasing function)

$$f(x) \geq f(a) = c, \quad \text{for } x \in I.$$

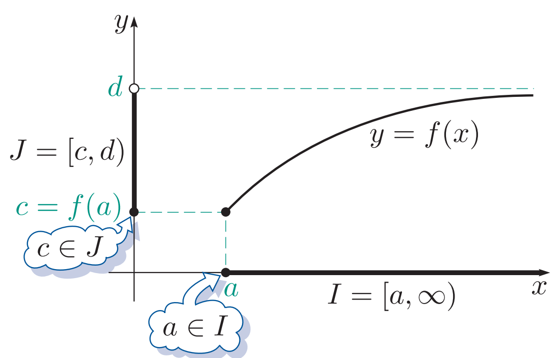


Figure 60 Step 3 of Strategy D15 when $a \in I$

Thus c is the corresponding left endpoint of J. This is illustrated in Figure 60.

If $a \notin I$, then let (a_n) be any decreasing sequence in I such that

$$a_n \to a \quad \text{as } n \to \infty. \tag{13}$$

Then $(f(a_n))$ is also a decreasing sequence (because f is an increasing function). Thus, by the Monotonic Sequence Theorem (Theorem D23 in Section 5 of Unit D2),

$$f(a_n) \to c \quad \text{as } n \to \infty, \tag{14}$$

where c is a real number or $-\infty$. This is illustrated in Figure 61.

We now prove that c is an endpoint of J and $c \notin J$.

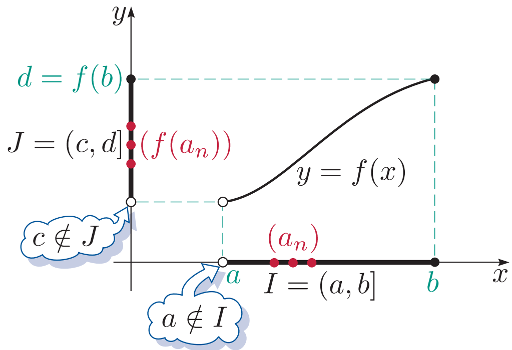


Figure 61 Step 3 of Strategy D15 when $a \notin I$

First we show that $(c, f(a_1)) \subset J$. If $c < y < f(a_1)$, then (by statement (14)) there exists n such that $f(a_n) < y < f(a_1)$, so $y = f(x)$ for some $x \in (a_n, a_1)$, by the Intermediate Value Theorem, and hence $y \in J$.

Finally, we show that $c \notin J$, using proof by contradiction. If $c \in J$, then $c = f(x)$ for some $x > a$, so (by statement (13)) there exists n such that $x > a_n$, which implies that $f(x) > f(a_n)$ and hence $f(x) > c$, a contradiction. ■

Summary

In this unit you have seen how to give a precise definition of what it means for a function to be continuous and how to use this to check whether or not a function is continuous at a point. You have also seen how to show that functions are continuous by using basic continuous functions together with rules such as the Combination Rules, the Composition Rule, the Squeeze Rule and the Glue Rule.

You have investigated some important properties of continuous functions and seen a number of applications of these properties. In particular, you have learnt how to use the Intermediate Value Theorem to locate zeros of continuous functions. Finally, you have seen how the Inverse Function Rule can be used to show that many functions have continuous inverse functions and to identify the domains of these inverse functions.

Continuity is one of the most important properties of functions and is used throughout analysis. You will meet it again when you study Book F *Analysis 2*.

Learning outcomes

After working through this unit, you should be able to:

- determine the domain and rule of the *sum*, *product*, *quotient* and *composite* of two real functions
- determine whether a given real function has an *inverse function*
- explain the meaning of the phrase 'f is *continuous* at a'
- use the Combination Rules, the Composition Rule, the Squeeze Rule and the Glue Rule for continuous functions
- recognise certain basic continuous functions
- state the Intermediate Value Theorem and use it to prove that certain equations have solutions
- determine an interval which contains all the *zeros* of a given polynomial
- state the Extreme Value Theorem and the Boundedness Theorem
- use the Inverse Function Rule to establish that a given function $f : I \longrightarrow J$ has a continuous inverse function $f^{-1} : J \longrightarrow I$
- define the inverse functions of certain standard functions
- define a^x for $a > 0$ and any $x \in \mathbb{R}$.

Solutions to exercises

Solution to Exercise D54

The domain of $f + g$ is $(-\pi/2, \pi/2)$; the rule is

$$(f + g)(x) = f(x) + g(x) = e^x + \tan x.$$

The domain of fg is $(-\pi/2, \pi/2)$; the rule is

$$(fg)(x) = f(x)g(x) = e^x \tan x.$$

The domain of f/g is $(-\pi/2, 0) \cup (0, \pi/2)$; the rule is

$$(f/g)(x) = f(x)/g(x) = e^x/\tan x = e^x \cot x.$$

Solution to Exercise D55

The domain of $f \circ g$ is

$$\begin{aligned}&\{x \in \mathbb{R} : \sin x \geq 0\}\\ &= \cdots \cup [-2\pi, -\pi] \cup [0, \pi] \cup [2\pi, 3\pi] \cup \cdots;\end{aligned}$$

the rule is

$$(f \circ g)(x) = \sqrt{\sin x}.$$

The domain of $g \circ f$ is

$$\{x \in [0, \infty) : \sqrt{x} \in \mathbb{R}\} = [0, \infty);$$

the rule is

$$(g \circ f)(x) = \sin \sqrt{x}.$$

Solution to Exercise D56

First we solve the equation

$$y = \frac{x+3}{x-2}$$

to obtain x in terms of y. We find that

$$y = \frac{x+3}{x-2} = 1 + \frac{5}{x-2} \iff x = 2 + \frac{5}{y-1}.$$

Thus f is one-to-one, so f has an inverse function with rule $f^{-1}(y) = 2 + 5/(y-1)$.

Now we find the image set of f, which is the domain of f^{-1}. For each $x \in (2, \infty)$, we have $5/(x-2) > 0$, so $y > 1$. Hence $f((2, \infty)) \subseteq (1, \infty)$.

Also, for each $y \in (1, \infty)$, we have $y - 1 > 0$, so

$$x = 2 + \frac{5}{y-1} \in (2, \infty).$$

Thus $f((2, \infty)) \supseteq (1, \infty)$, so

$$f((2, \infty)) = (1, \infty).$$

Hence the domain of f^{-1} is $(1, \infty)$ so, adopting the usual practice of denoting the domain variable by x, we have

$$f^{-1}(x) = 2 + \frac{5}{x-1} \quad (x \in (1, \infty)),$$

which may also be written as

$$f^{-1}(x) = \frac{2x+3}{x-1} \quad (x \in (1, \infty)).$$

Solution to Exercise D57

(a) If $0 \leq x_1 < x_2$, then $2x_1 < 2x_2$ and $x_1^4 < x_2^4$. Hence

$$x_1^4 + 2x_1 + 3 < x_2^4 + 2x_2 + 3,$$

so f is strictly increasing, and thus one-to-one.

(b) If $0 < x_1 < x_2$, then $1/x_1 > 1/x_2$ and $x_1^2 < x_2^2$, so $-x_1^2 > -x_2^2$. Hence

$$\frac{1}{x_1} - x_1^2 > \frac{1}{x_2} - x_2^2,$$

so f is strictly decreasing, and thus one-to-one.

Solution to Exercise D58

(a) $\lim\limits_{n\to\infty} 3x_n = 6$, by the Multiple Rule for sequences.

(b) $\lim\limits_{n\to\infty} x_n^2 = 4$, by the Product Rule for sequences.

(c) $\lim\limits_{n\to\infty} 1/x_n = 1/2$, by the Quotient Rule for sequences.

Solution to Exercise D59

(a) We guess that f is continuous at $a = 2$. The domain of f is $\mathbb{R}$. If (x_n) is a sequence in $\mathbb{R}$ with $x_n \to 2$, then

$$f(x_n) = x_n^3 - 2x_n^2 \to 8 - 8 = 0 = f(2),$$

by the Combination Rules for sequences.

Hence f is continuous at $a = 2$.

(b) We guess that $f(x) = \lfloor x \rfloor$ is discontinuous at $a = 1$, since

$$f(x) = 0, \quad \text{for } 0 \leq x < 1,$$

and $f(1) = 1$. We choose

$$x_n = 1 - \frac{1}{n}, \quad n = 1, 2, \ldots.$$

Then $x_n \to 1$ but $f(x_n) = 0$ for $n = 1, 2, \ldots$, so

$$f(x_n) \to 0 \neq 1 = f(1).$$

Hence f is discontinuous at $a = 1$.

Solution to Exercise D60

(a) Let $a \in \mathbb{R}$ and let (x_n) be a sequence in $\mathbb{R}$ with $x_n \to a$. Then

$$f(x_n) = 1 \to 1 = f(a).$$

Hence f is continuous.

(b) Let $a \in \mathbb{R}$ and let (x_n) be a sequence in $\mathbb{R}$ with $x_n \to a$. Then

$$f(x_n) = x_n \to a = f(a).$$

Hence f is continuous.

Solution to Exercise D61

We guess that f is continuous on $[0, \infty)$.

If we let $g(x) = x^5$ and $h(x) = |x|$, so that $h(g(x)) = \left|x^5\right|$, we can express f as a composite function $f = h \circ g$. But g is continuous (on $\mathbb{R}$), since it is a polynomial, and h is continuous on $\mathbb{R}$ by Worked Exercise D45. It then follows from the Composition Rule that $f = h \circ g$ is continuous on $\mathbb{R}$.

Solution to Exercise D62

The function $x \longmapsto x^2 + 2x + 2$ is a polynomial, and so is continuous on $\mathbb{R}$. The function $x \longmapsto \sqrt{x}$ is continuous on $[0, \infty)$ (as we saw in Worked Exercise D46) and $x^2 + 2x + 2 = (x+1)^2 + 1 > 0$ for all $x \in \mathbb{R}$; so, by the Composition Rule for continuous functions, the function $x \longmapsto \sqrt{x^2 + 2x + 2}$ is continuous on $\mathbb{R}$.

Next, the function $x \longmapsto \dfrac{-3x}{x^4 + 4}$ is a rational function; hence it is continuous on its domain, which is $\mathbb{R}$ since $x^4 + 4 \neq 0$.

Then, by the Sum Rule for continuous functions, the function

$$x \longmapsto \sqrt{x^2 + 2x + 2} - \frac{3x}{x^4 + 4} \quad (= f(x))$$

is continuous on $\mathbb{R}$.

Solution to Exercise D63

(a) We prove that

$$f(x) = \begin{cases} x^2 \cos(1/x^2), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is continuous at 0, using the Squeeze Rule.

Since

$$-1 \leq \cos(1/x^2) \leq 1, \quad \text{for } x \neq 0,$$

and $x^2 \geq 0$, we have

$$-x^2 \leq x^2 \cos(1/x^2) \leq x^2, \quad \text{for } x \neq 0.$$

Since $f(0) = 0$, we deduce that

$$-x^2 \leq f(x) \leq x^2, \quad \text{for } x \in \mathbb{R}.$$

Thus if we take $I = \mathbb{R}$, with

$$g(x) = -x^2 \quad \text{and} \quad h(x) = x^2,$$

then

$$g(x) \leq f(x) \leq h(x), \quad \text{for } x \in I,$$

so condition 1 of the Squeeze Rule holds.

Next, $f(0) = g(0) = h(0) = 0$, so condition 2 of the Squeeze Rule is satisfied.

Finally, the functions g and h are polynomials, and so they are continuous at 0. Thus condition 3 of the Squeeze Rule is satisfied.

Hence f is continuous at 0, by the Squeeze Rule.

(b) We prove that

$$f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

is discontinuous at 0.

According to Strategy D14, we have to find *one* sequence (x_n) such that

$$x_n \to 0 \quad \text{but} \quad f(x_n) \not\to f(0) = 0.$$

We use the fact that $\sin\left(2n+\frac{1}{2}\right)\pi = 1$, for $n = 0, 1, 2, \ldots$, and choose

$$x_n = \frac{1}{\left(2n+\frac{1}{2}\right)\pi}, \quad n = 0, 1, 2, \ldots.$$

Then $x_n \to 0$ and

$$\begin{aligned} f(x_n) &= \sin(1/x_n) \\ &= \sin\left(2n+\tfrac{1}{2}\right)\pi = 1, \quad \text{for } n = 0, 1, 2, \ldots, \end{aligned}$$

so

$$f(x_n) \nrightarrow f(0) = 0.$$

Hence f is discontinuous at 0.

Solution to Exercise D64

We use the Glue Rule.

Let $I = \mathbb{R}$ and define the functions

$$g(x) = x^3 - 3x + 5 \quad \text{and} \quad h(x) = \frac{2x+1}{3x-2}.$$

Then f is defined on I and $1 \in I$. Also,

$$f(x) = g(x), \quad \text{for } x \in (-\infty, 1),$$

and

$$f(x) = h(x), \quad \text{for } x \in (1, \infty),$$

so condition 1 of the Glue Rule holds with $a = 1$.

Moreover, $f(1) = g(1) = h(1) = 3$, so condition 2 holds.

Finally, g and h are continuous functions, being a polynomial and a rational function respectively, and so they are both continuous at 1 since the only zero of the denominator of h is $\frac{2}{3}$. Thus condition 3 holds.

Hence f is continuous at 1, by the Glue Rule.

Solution to Exercise D65

Let $g(x) = \sin x$, $h(x) = \sqrt{x}$, and $k(x) = x^2 + 1$. Then g is continuous (by Theorem D47), h is continuous (by Worked Exercise D46), and k is continuous (being a polynomial); so

$$g(h(k(x))) = \sin\left(\sqrt{x^2+1}\right)$$

is continuous, by the Composition Rule (applied twice). Hence

$$f(x) = k(x) + 3g(h(k(x)))$$

is continuous, by the Combination Rules.

Solution to Exercise D66

Let $g(x) = -x^2$, $h(x) = e^x$ and $k(x) = x^5 - 5x^2$. Then g and k are both continuous as they are polynomials, and $h(x)$ is continuous by Theorem D50. Hence, by the Composition Rule and the Multiple Rule,

$$7h(g(x)) = 7e^{-x^2}$$

is continuous. Finally,

$$f(x) = k(x) + 7h(g(x))$$

is continuous by the Sum Rule.

Solution to Exercise D67

We know that c lies in $\left(\frac{1}{2}, 1\right)$, so we calculate

$$f\left(\tfrac{3}{4}\right) = \left(\tfrac{3}{4}\right)^5 + \tfrac{3}{4} - 1 \approx -0.0127 < 0.$$

Thus c lies in $\left(\frac{3}{4}, 1\right)$, so we calculate

$$f\left(\tfrac{7}{8}\right) = \left(\tfrac{7}{8}\right)^5 + \tfrac{7}{8} - 1 \approx 0.388 > 0.$$

Thus c lies in $\left(\frac{3}{4}, \frac{7}{8}\right)$, so we calculate

$$f\left(\tfrac{13}{16}\right) = \left(\tfrac{13}{16}\right)^5 + \tfrac{13}{16} - 1 \approx 0.167 > 0.$$

Thus c lies in $\left(\frac{3}{4}, \frac{13}{16}\right)$, an interval of length $\frac{1}{16}$.

Solution to Exercise D68

If $f(0) = 0$ or $f(1) = 1$, then we can take $c = 0$ or $c = 1$, respectively.

Otherwise, we have $f(0) > 0$ and $f(1) < 1$, since $0 \le f(x) \le 1$, for $0 \le x \le 1$.

We consider the function

$$g(x) = f(x) - x \quad (x \in [0, 1])$$

and show that g has a zero c in $(0, 1)$.

Now g is continuous on $[0, 1]$, by the Combination Rules. Moreover,

$$g(0) = f(0) - 0 > 0$$

and

$$g(1) = f(1) - 1 < 0.$$

Thus, by the Intermediate Value Theorem, there is a number c in $(0,1)$ such that

$$g(c) = 0, \quad \text{so} \quad f(c) = c.$$

Solution to Exercise D69

We have

$$p(-1) = -3,\ p(0) = 1,\ p(1) = -1,\ p(2) = 3,$$

so

$$\begin{aligned} &p(-1) < 0 < p(0),\\ &p(0) > 0 > p(1),\\ &p(1) < 0 < p(2). \end{aligned}$$

Since p is continuous, we deduce by the Intermediate Value Theorem that p has a zero in each of the intervals

$$(-1,0), \quad (0,1), \quad (1,2).$$

Solution to Exercise D70

When

$$p(x) = x^5 + 3x^4 - x - 1 \quad (x \in \mathbb{R}),$$

we have

$$M = 1 + \max\{|3|, |-1|, |-1|\} = 4,$$

so all the zeros of p lie in $(-4,4)$, by Theorem D54.

Calculating $p(n)$ for integers n in $[-4,4]$, we obtain

n	-4	-3	-2	-1	0	1	2
$p(n)$	-253	2	17	2	-1	2	77

Thus p changes sign on each of the intervals

$$[-4,-3], \quad [-1,0], \quad [0,1].$$

Since p is continuous, we deduce by the Intermediate Value Theorem that p has a zero in each of the intervals

$$(-4,-3), \quad (-1,0), \quad (0,1).$$

Thus p has at least three zeros.

Solution to Exercise D71

We use Strategy D15.

1. We showed that $-f$ is strictly decreasing on $(0,\infty)$ in Exercise D57(b), and so f is strictly increasing.

2. The function

$$f(x) = x^2 - \frac{1}{x} = \frac{x^3 - 1}{x} \quad (x \in (0,\infty))$$

is the restriction to $(0,\infty)$ of a rational function which is continuous on $\mathbb{R} - \{0\}$. Hence f is continuous.

3. Now choose the increasing sequence (n), which tends to ∞, the right endpoint of $(0,\infty)$. Then

$$f(n) = n^2 - 1/n \to \infty \ \text{ as } n \to \infty,$$

by the Reciprocal Rule. Thus the right endpoint of $J = f((0,\infty))$ is ∞.

Then choose the decreasing sequence $(1/n)$, which tends to 0, the left endpoint of $(0,\infty)$. Then

$$f(1/n) = 1/n^2 - n \to -\infty \ \text{ as } n \to \infty,$$

by the Reciprocal Rule. Thus the left endpoint of $J = f((0,\infty))$ is $-\infty$.

Hence $J = (-\infty,\infty) = \mathbb{R}$, so f has a continuous inverse function

$$f^{-1} : \mathbb{R} \longrightarrow (0,\infty),$$

by the Inverse Function Rule.

Solution to Exercise D72

(a) Since $\sin(\pi/4) = 1/\sqrt{2}$ and $\pi/4$ lies in $[-\pi/2,\pi/2]$, we have

$$\sin^{-1}(1/\sqrt{2}) = \frac{\pi}{4}.$$

Since $\cos(2\pi/3) = -\frac{1}{2}$ and $2\pi/3$ lies in $[0,\pi]$,

$$\cos^{-1}(-\tfrac{1}{2}) = \frac{2\pi}{3}.$$

Since $\tan(\pi/3) = \sqrt{3}$ and $\pi/3$ lies in $(-\pi/2,\pi/2)$,

$$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}.$$

(b) Following the hint, we put $y = \sin^{-1} x$. Then

$$\cos(2\sin^{-1} x) = \cos(2y) = 1 - 2\sin^2 y = 1 - 2x^2,$$

since $x = \sin y$.

Solution to Exercise D73

Following the hint, we put $a = \log x$ and $b = \log y$. Then $x = e^a$ and $y = e^b$, so

$$\begin{aligned}\log(xy) &= \log(e^a e^b)\\ &= \log(e^{a+b}) = a + b = \log x + \log y.\end{aligned}$$

Solution to Exercise D74

Let $y = \cosh^{-1} x$, where $x \geq 1$. Then

$$x = \cosh y = \tfrac{1}{2}(e^y + e^{-y}).$$

Hence, by multiplying both sides by e^y,

$$e^{2y} - 2xe^y + 1 = (e^y)^2 - 2xe^y + 1 = 0.$$

This is a quadratic equation in e^y, with solutions

$$e^y = x \pm \sqrt{x^2 - 1}.$$

Both choices of $\pm$ give a positive expression on the right, but we also have $e^y \geq 1$, since $y = \cosh^{-1} x \geq 0$.

Since

$$\left(x + \sqrt{x^2 - 1}\right)\left(x - \sqrt{x^2 - 1}\right) = x^2 - (x^2 - 1) = 1$$

and $x + \sqrt{x^2 - 1} \geq 1$, we have $x - \sqrt{x^2 - 1} \leq 1$.

Thus we choose the $+$ sign, to give

$$y = \cosh^{-1} x = \log\left(x + \sqrt{x^2 - 1}\right).$$

(The value $y = \log\left(x - \sqrt{x^2 - 1}\right)$ gives the negative solution of the equation $\cosh y = x$.)

Solution to Exercise D75

(a) For $x > 0$, we have

$$f(x) = x^\alpha = e^{\alpha \log x}.$$

Now the functions

$$\begin{aligned}&x \longmapsto \log x \quad (x \in (0, \infty)),\\ &x \longmapsto e^x \quad (x \in \mathbb{R}),\end{aligned}$$

are both continuous, so f is continuous by the Multiple Rule and the Composition Rule.

(b) For $x > 0$, we have

$$f(x) = x^x = e^{x \log x}.$$

Now the functions

$$\begin{aligned}&x \longmapsto \log x \quad (x \in (0, \infty)),\\ &x \longmapsto x \quad (x \in \mathbb{R}),\\ &x \longmapsto e^x \quad (x \in \mathbb{R}),\end{aligned}$$

are all continuous, so f is continuous by the Product Rule and the Composition Rule.